8 Vector Analysis in Higher Dimensions

Introduction

In this concluding chapter, our goal is to find a way to unify and extend the three main theorems of vector analysis (namely, the theorems of Green, Gauss, and Stokes). To accomplish such a task, we need to develop the notion of a **differential form** whose integral embraces and generalizes line, surface, and volume integrals.

8.1 An Introduction to Differential Forms

Throughout this section, U will denote an open set in $\mathbf{R}^n$, where $\mathbf{R}^n$ has coordinates $(x_1, x_2, \ldots, x_n)$, as usual. Any functions that appear are assumed to be appropriately differentiable.

Differential Forms

We begin by giving a new name to an old friend. If $f\colon U \subseteq \mathbf{R}^n \to \mathbf{R}$ is a scalar-valued function (of class C^k), we will also refer to f as a **differential 0-form**, or just a **0-form** for short. 0-forms can be added to one another and multiplied together, as well we know.

The next step is to describe differential 1-forms. Ultimately, we will see that a differential 1-form is a generalization of $f(x)\,dx$—that is, of something that can be integrated with respect to a single variable, such as with a line integral. More precisely, in $\mathbf{R}^n$, the **basic differential 1-forms** are denoted $dx_1, dx_2, \ldots, dx_n$. A general **(differential) 1-form** ω is an expression that is built from the basic 1-forms as

$$\omega = F_1(x_1, \ldots, x_n)\,dx_1 + F_2(x_1, \ldots, x_n)\,dx_2 + \ldots + F_n(x_1, \ldots, x_n)\,dx_n,$$

where, for $j = 1, \ldots, n$, F_j is a scalar-valued function (of class C^k) on $U \subseteq \mathbf{R}^n$. Differential 1-forms can be added to one another, and we can multiply a 0-form f and a 1-form ω (both defined on $U \subseteq \mathbf{R}^n$) in the obvious way: If

$$\omega = F_1\,dx_1 + F_2\,dx_2 + \cdots + F_n\,dx_n,$$

then

$$f\omega = fF_1\,dx_1 + fF_2\,dx_2 + \cdots + fF_n\,dx_n.$$

EXAMPLE 1 In $\mathbf{R}^3$, let

$$\omega = xyz\,dx + z^2 \cos y\,dy + ze^x\,dz \quad \text{and} \quad \eta = (y - z)\,dx + z^2 \sin y\,dy - 2\,dz.$$

Then

$$\omega + \eta = (xyz + y - z)\,dx + z^2(\cos y + \sin y)\,dy + (ze^x - 2)\,dz.$$

If $f(x, y, z) = xe^y - z$, then

$$f\omega = (xe^y - z)xyz\,dx + (xe^y - z)z^2 \cos y\,dy + (xe^y - z)ze^x dz.$$ ◆

Thus far, we have described 1-forms merely as formal expressions in certain symbols. But 1-forms can also be thought of as functions. The basic 1-forms $dx_1, \ldots, dx_n$ take as argument a vector $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ in $\mathbf{R}^n$; the value of dx_i on $\mathbf{a}$ is

$$dx_i(\mathbf{a}) = a_i.$$

In others words, dx_i extracts the ith component of the vector $\mathbf{a}$.

More generally, for each $\mathbf{x}_0 \in U$, the 1-form ω gives rise to a combination $\omega_{\mathbf{x}_0}$ of basic 1-forms

$$\omega_{\mathbf{x}_0} = F_1(\mathbf{x}_0)\,dx_1 + \cdots + F_n(\mathbf{x}_0)\,dx_n;$$

$\omega_{\mathbf{x}_0}$ acts on the vector $\mathbf{a} \in \mathbf{R}^n$ as

$$\omega_{\mathbf{x}_0}(\mathbf{a}) = F_1(\mathbf{x}_0)\,dx_1(\mathbf{a}) + F_2(\mathbf{x}_0)\,dx_2(\mathbf{a}) + \cdots + F_n(\mathbf{x}_0)\,dx_n(\mathbf{a}).$$

EXAMPLE 2 Suppose ω is the 1-form defined on $\mathbf{R}^3$ by

$$\omega = x^2yz\,dx + y^2z\,dy - 3xyz\,dz.$$

If $\mathbf{x}_0 = (1, -2, 5)$ and $\mathbf{a} = (a_1, a_2, a_3)$, then

$$\begin{aligned}\omega_{(1,-2,5)}(\mathbf{a}) &= -10\,dx(\mathbf{a}) + 20\,dy(\mathbf{a}) + 30\,dz(\mathbf{a})\\ &= -10a_1 + 20a_2 + 30a_3,\end{aligned}$$

and, if $\mathbf{x}_0 = (3, 4, 6)$, then

$$\begin{aligned}\omega_{(3,4,6)}(\mathbf{a}) &= 216\,dx(\mathbf{a}) + 96\,dy(\mathbf{a}) - 216\,dz(\mathbf{a})\\ &= 216a_1 + 96a_2 - 216a_3.\end{aligned}$$

The notation suggests that a 1-form is a function of the vector $\mathbf{a}$ but that this function varies from point to point as $\mathbf{x}_0$ changes. Indeed, 1-forms are actually functions on vector fields. ◆

A **basic (differential) 2-form** on $\mathbf{R}^n$ is an expression of the form

$$dx_i \wedge dx_j, \quad i, j = 1, \ldots, n.$$

It is also a function that requires *two* vector arguments $\mathbf{a}$ and $\mathbf{b}$, and we evaluate this function as

$$dx_i \wedge dx_j(\mathbf{a}, \mathbf{b}) = \begin{vmatrix} dx_i(\mathbf{a}) & dx_i(\mathbf{b}) \\ dx_j(\mathbf{a}) & dx_j(\mathbf{b}) \end{vmatrix}.$$

(The determinant represents, up to sign, the area of the parallelogram spanned by the projections of $\mathbf{a}$ and $\mathbf{b}$ in the x_ix_j-plane.) It is not difficult to see that, for $i, j = 1, \ldots, n$,

$$dx_i \wedge dx_j = -dx_j \wedge dx_i \tag{1}$$

and

$$dx_i \wedge dx_i = 0. \tag{2}$$

Formula (1) can be established by comparing $dx_i \wedge dx_j(\mathbf{a}, \mathbf{b})$ with $dx_j \wedge dx_i(\mathbf{a}, \mathbf{b})$. Formula (2) follows from formula (1). Given formulas (1) and (2), we see that we can generate all the linearly independent, nontrivial basic 2-forms on $\mathbf{R}^n$ by listing all possible terms $dx_i \wedge dx_j$, where i and j are integers between 1 and n with $i < j$:

$$\begin{gathered} dx_1 \wedge dx_2,\ dx_1 \wedge dx_3, \ldots, dx_1 \wedge dx_n, \\ dx_2 \wedge dx_3, \ldots, dx_2 \wedge dx_n, \\ \vdots \\ dx_{n-1} \wedge dx_n. \end{gathered}$$

To count how many 2-forms are in this list, note that there are n choices for dx_i and $n-1$ choices for dx_j (so that $dx_i \neq dx_j$ in view of (2)), and a "correction" factor of 2 so as not to count both $dx_i \wedge dx_j$ and $dx_j \wedge dx_i$ in light of (1). Hence, there are $n(n-1)/2$ independent 2-forms.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. A general (**differential**) **2-form** on $U \subseteq \mathbf{R}^n$ is an expression

$$\omega = F_{12}(\mathbf{x})\, dx_1 \wedge dx_2 + F_{13}(\mathbf{x})\, dx_1 \wedge dx_3 + \cdots + F_{n-1n}(\mathbf{x})\, dx_{n-1} \wedge dx_n,$$

where each F_{ij} is a real-valued function $F_{ij}\colon U \subseteq \mathbf{R}^n \to \mathbf{R}$. The idea here is to generalize something that can be integrated with respect to two variables—such as with a surface integral.

EXAMPLE 3 In $\mathbf{R}^3$, a general 2-form may be written as

$$F_1(x, y, z)\, dy \wedge dz + F_2(x, y, z)\, dz \wedge dx + F_3(x, y, z)\, dx \wedge dy.$$

The reason for using this somewhat curious ordering of the terms in the sum will, we hope, become clear later in the chapter. ◆

Given a point $\mathbf{x}_0 \in U \subseteq \mathbf{R}^n$, to evaluate a general 2-form on the ordered pair $(\mathbf{a}, \mathbf{b})$ of vectors, we have

$$\begin{aligned} \omega_{\mathbf{x}_0}(\mathbf{a}, \mathbf{b}) = {} & F_{12}(\mathbf{x}_0)\, dx_1 \wedge dx_2(\mathbf{a}, \mathbf{b}) + F_{13}(\mathbf{x}_0)\, dx_1 \wedge dx_3(\mathbf{a}, \mathbf{b}) \\ & + \cdots + F_{n-1n}(\mathbf{x}_0)\, dx_{n-1} \wedge dx_n(\mathbf{a}, \mathbf{b}). \end{aligned}$$

EXAMPLE 4 In $\mathbf{R}^3$, let $\omega = 3xy\, dy \wedge dz + (2y + z)\, dz \wedge dx + (x - z)\, dx \wedge dy$. Then

$$\begin{aligned} \omega_{(1,2,-3)}(\mathbf{a}, \mathbf{b}) &= 6\, dy \wedge dz(\mathbf{a}, \mathbf{b}) + dz \wedge dx(\mathbf{a}, \mathbf{b}) + 4\, dx \wedge dy(\mathbf{a}, \mathbf{b}) \\ &= 6 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} + \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} + 4 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \\ &= 6(a_2 b_3 - a_3 b_2) + (a_3 b_1 - a_1 b_3) + 4(a_1 b_2 - a_2 b_1). \end{aligned}$$

◆

Finally, we generalize the notions of 1-forms and 2-forms to provide a definition of a ***k*-form**.

DEFINITION 1.1 Let k be a positive integer. A **basic (differential) *k*-form** on $\mathbf{R}^n$ is an expression of the form

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k},$$

where $1 \leq i_j \leq n$ for $j = 1, \ldots, k$. The basic k-forms are also functions that require k vector arguments $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ and are evaluated as

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}(\mathbf{a}_1, \ldots, \mathbf{a}_k) = \det \begin{bmatrix} dx_{i_1}(\mathbf{a}_1) & dx_{i_1}(\mathbf{a}_2) & \cdots & dx_{i_1}(\mathbf{a}_k) \\ dx_{i_2}(\mathbf{a}_1) & dx_{i_2}(\mathbf{a}_2) & \cdots & dx_{i_2}(\mathbf{a}_k) \\ \vdots & \vdots & \ddots & \vdots \\ dx_{i_k}(\mathbf{a}_1) & dx_{i_k}(\mathbf{a}_2) & \cdots & dx_{i_k}(\mathbf{a}_k) \end{bmatrix}.$$

EXAMPLE 5 Let

$$\mathbf{a}_1 = (1, 2, -1, 3, 0), \quad \mathbf{a}_2 = (5, 4, 3, 2, 1), \quad \text{and} \quad \mathbf{a}_3 = (0, 1, 3, -2, 0)$$

be three vectors in $\mathbf{R}^5$. Then we have

$$dx_1 \wedge dx_3 \wedge dx_5(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \det \begin{bmatrix} 1 & 5 & 0 \\ -1 & 3 & 3 \\ 0 & 1 & 0 \end{bmatrix} = -3.$$ ◆

Using properties of determinants, we can show that

$$dx_{i_1} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_l} \wedge \cdots \wedge dx_{i_k}$$
$$= -dx_{i_1} \wedge \cdots \wedge dx_{i_l} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_k} \tag{3}$$

and

$$dx_{i_1} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_j} \wedge \ldots \wedge dx_{i_k} = 0. \tag{4}$$

Formula (3) says that switching two terms (namely, dx_{i_j} and dx_{i_l}) in the basic k-form $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ causes a sign change, and formula (4) says that a basic k-form containing two identical terms is zero. Formulas (3) and (4) generalize formulas (1) and (2).

DEFINITION 1.2 A general **(differential) *k*-form** on $U \subseteq \mathbf{R}^n$ is an expression of the form

$$\omega = \sum_{i_1, \ldots, i_k = 1}^{n} F_{i_1 \ldots i_k}(\mathbf{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where each $F_{i_1 \ldots i_k}$ is a real-valued function $F_{i_1 \ldots i_k} \colon U \to \mathbf{R}$. Given a point $\mathbf{x}_0 \in U$, we evaluate ω on an ordered k-tuple $(\mathbf{a}_1, \ldots, \mathbf{a}_k)$ of vectors as

$$\omega_{\mathbf{x}_0}(\mathbf{a}_1, \ldots, \mathbf{a}_k) = \sum_{i_1, \ldots, i_k = 1}^{n} F_{i_1 \ldots i_k}(\mathbf{x}_0)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}(\mathbf{a}_1, \ldots, \mathbf{a}_k).$$

Note that a 0-form is so named because, in order to be consistent with a 1-form or 2-form, it must take zero vector arguments!

In view of formulas (3) and (4), we write a general k-form as

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} F_{i_1 \dots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

(That is, the sum may be taken over strictly increasing indices $i_1, \dots, i_k$.) For example, the 4-form

$$\omega = x_2\, dx_1 \wedge dx_3 \wedge dx_4 \wedge dx_5 + (x_3 - x_5^2)\, dx_1 \wedge dx_2 \wedge dx_5 \wedge dx_3 + x_1 x_3\, dx_5 \wedge dx_3 \wedge dx_4 \wedge dx_1$$

may be written in the "standard form" with increasing indices as

$$\omega = (x_2 - x_1 x_3)\, dx_1 \wedge dx_3 \wedge dx_4 \wedge dx_5 + (x_5^2 - x_3)\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5.$$

Two k-forms may be added in the obvious way, and the product of a 0-form f and a k-form ω is analogous to the product of a 0-form and a 1-form.

Exterior Product

The symbol $\wedge$ that we have been using does, in fact, denote a type of multiplication called the **exterior** (or **wedge**) **product**. The exterior product can be extended to general differential forms in the following manner:

DEFINITION 1.3 Let $U \subseteq \mathbf{R}^n$ be open. Let f denote a 0-form on U. Let $\omega = \sum F_{i_1 \dots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ denote a k-form on U and $\eta = \sum G_{j_1 \dots j_l} dx_{j_1} \wedge \cdots \wedge dx_{j_l}$ an l-form. Then we define

$$f \wedge \omega = f\omega = \sum f F_{i_1 \dots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

$$\omega \wedge \eta = \sum F_{i_1 \dots i_k} G_{j_1 \dots j_l} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}.$$

Thus, the wedge product of a k-form and an l-form is a $(k + l)$-form.

EXAMPLE 6 Let

$$\omega = x_1^2\, dx_1 \wedge dx_2 + (2x_3 - x_2)\, dx_1 \wedge dx_3 + e^{x_3}\, dx_3 \wedge dx_4$$

and

$$\eta = x_4\, dx_1 \wedge dx_3 \wedge dx_5 + x_6\, dx_2 \wedge dx_4 \wedge dx_6$$

be, respectively, a 2-form and a 3-form on $\mathbf{R}^6$. Then Definition 1.3 yields

$$\begin{aligned} \omega \wedge \eta = {} & x_1^2 x_4\, dx_1 \wedge dx_2 \wedge dx_1 \wedge dx_3 \wedge dx_5 \\ & + (2x_3 - x_2) x_4\, dx_1 \wedge dx_3 \wedge dx_1 \wedge dx_3 \wedge dx_5 \\ & + e^{x_3} x_4\, dx_3 \wedge dx_4 \wedge dx_1 \wedge dx_3 \wedge dx_5 \\ & + x_1^2 x_6\, dx_1 \wedge dx_2 \wedge dx_2 \wedge dx_4 \wedge dx_6 \\ & + (2x_3 - x_2) x_6\, dx_1 \wedge dx_3 \wedge dx_2 \wedge dx_4 \wedge dx_6 \\ & + e^{x_3} x_6\, dx_3 \wedge dx_4 \wedge dx_2 \wedge dx_4 \wedge dx_6. \end{aligned}$$

Because of formula (4), most of the terms in this sum are zero. In fact,

$$\begin{aligned}\omega \wedge \eta &= (2x_3 - x_2)x_6\, dx_1 \wedge dx_3 \wedge dx_2 \wedge dx_4 \wedge dx_6 \\ &= (x_2 - 2x_3)x_6\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_6,\end{aligned}$$

using formula (3). ◆

From the various definitions and observations made so far, we can establish the following results, which are useful when computing with differential forms:

PROPOSITION 1.4 (Properties of the exterior product) Assume that all the differential forms that follow are defined on $U \subseteq \mathbf{R}^n$:

1. **Distributivity.** If ω_1 and ω_2 are k-forms and η is an l-form, then
$$(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta.$$
2. **Anticommutativity.** If ω is a k-form and η an l-form, then
$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega.$$
3. **Associativity.** If ω is a k-form, η an l-form, and τ a p-form, then
$$(\omega \wedge \eta) \wedge \tau = \omega \wedge (\eta \wedge \tau).$$
4. **Homogeneity.** If ω is a k-form, η an l-form, and f a 0-form, then
$$(f\omega) \wedge \eta = f(\omega \wedge \eta) = \omega \wedge (f\eta).$$

8.1 Exercises

Determine the values of the following differential forms on the ordered sets of vectors indicated in Exercises 1–7.

1. $dx_1 - 3\,dx_2$; $\mathbf{a} = (7, 3)$

2. $2\,dx + 6\,dy - 5\,dz$; $\mathbf{a} = (1, -1, -2)$

3. $3\,dx_1 \wedge dx_2$; $\mathbf{a} = (4, -1)$, $\mathbf{b} = (2, 0)$

4. $4\,dx \wedge dy - 7\,dy \wedge dz$; $\mathbf{a} = (0, 1, -1)$, $\mathbf{b} = (1, 3, 2)$

5. $7\,dx \wedge dy \wedge dz$; $\mathbf{a} = (1, 0, 3)$, $\mathbf{b} = (2, -1, 0)$, $\mathbf{c} = (5, 2, 1)$

6. $dx_1 \wedge dx_2 + 2\,dx_2 \wedge dx_3 + 3\,dx_3 \wedge dx_4$; $\mathbf{a} = (1, 2, 3, 4)$, $\mathbf{b} = (4, 3, 2, 1)$

7. $2\,dx_1 \wedge dx_3 \wedge dx_4 + dx_2 \wedge dx_3 \wedge dx_5$; $\mathbf{a} = (1, 0, -1, 4, 2)$, $\mathbf{b} = (0, 0, 9, 1, -1)$, $\mathbf{c} = (5, 0, 0, 0, -2)$

8. Let ω be the 1-form on $\mathbf{R}^3$ defined by

$$\omega = x^2 y\, dx + y^2 z\, dy + z^3 x\, dz.$$

Find $\omega_{(3,-1,4)}(\mathbf{a})$, where $\mathbf{a} = (a_1, a_2, a_3)$.

9. Let ω be the 2-form on $\mathbf{R}^4$ given by

$$\omega = x_1 x_3\, dx_1 \wedge dx_3 - x_2 x_4\, dx_2 \wedge dx_4.$$

Find $\omega_{(2,-1,-3,1)}(\mathbf{a}, \mathbf{b})$.

10. Let ω be the 2-form on $\mathbf{R}^3$ given by

$$\omega = \cos x\, dx \wedge dy - \sin z\, dy \wedge dz + (y^2 + 3) dx \wedge dz.$$

Find $\omega_{(0,-1,\pi/2)}(\mathbf{a}, \mathbf{b})$, where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$.

11. Let ω be as in Exercise 10. Find $\omega_{(x,y,z)}((2, 0, -1), (1, 7, 5))$.

12. Let ω be the 3-form on $\mathbf{R}^3$ given by

$$\omega = \left(e^x \cos y + (y^2 + 2)e^{2z}\right) dx \wedge dy \wedge dz.$$

Find $\omega_{(0,0,0)}(\mathbf{a}, \mathbf{b}, \mathbf{c})$, where $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, and $\mathbf{c} = (c_1, c_2, c_3)$.

13. Let ω be as in Exercise 12. Find $\omega_{(x,y,z)}((1, 0, 0), (0, 2, 0), (0, 0, 3))$.

In Exercises 14–19, determine $\omega \wedge \eta$.

14. On $\mathbf{R}^3$: $\omega = 3\,dx + 2\,dy - x\,dz$; $\eta = x^2 dx - \cos y\, dy + 7\,dz$.

15. On $\mathbf{R}^3$: $\omega = y\,dx - x\,dy$; $\eta = z\,dx \wedge dy + y\,dx \wedge dz + x\,dy \wedge dz$.