70. Verify Eq. (2).

Computer Grapher

If you have access to a 3-D computer grapher, try graphing the surfaces in Exercises 71–77.

71. $z = y^2$ **72.** $z = 1 - y^2$

73. $z = x^2 + y^2$ **74.** $z = x^2 + 2y^2$

75. $z = \sqrt{1 - x^2}$ (upper half of a circular cylinder)

76. $z = \sqrt{1 - (y^2/4)}$ (upper half of an elliptical cylinder)

77. $z = \sqrt{x^2 + 2y^2 + 4}$ (one sheet of an elliptic hyperboloid)

EXPLORER PROGRAM

3D Grapher — Graphs equations of the form $z = f(x, y)$

10.8 Cylindrical and Spherical Coordinates

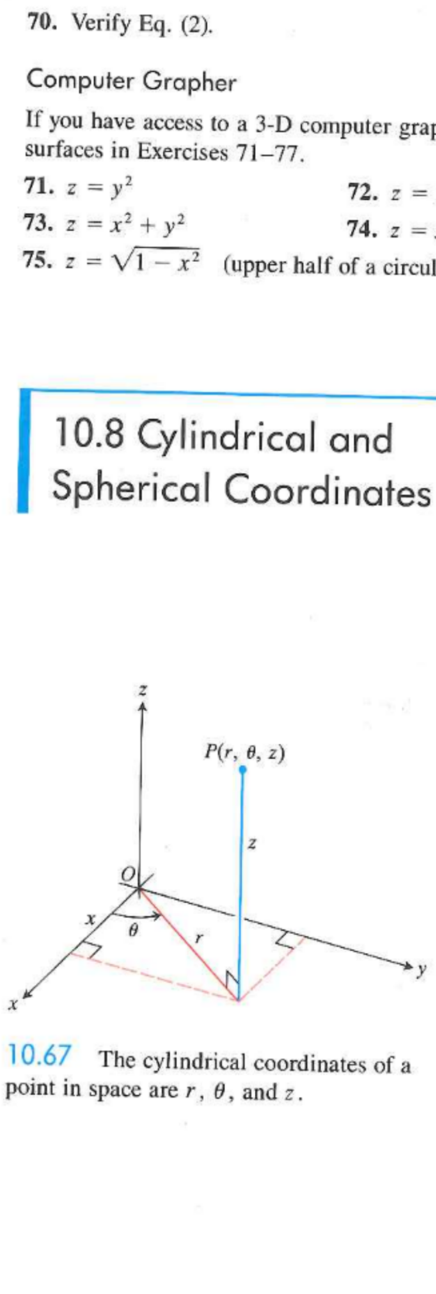
In this section we introduce two new systems of coordinates for space: the cylindrical coordinate system and the spherical coordinate system. In the cylindrical coordinate system, cylinders whose axes lie along the z-axis and planes that contain the z-axis have especially simple equations. In the spherical coordinate system, spheres centered at the origin and single cones at the origin whose axes lie along the z-axis have especially simple equations. When your work involves these shapes, these may be the best coordinate systems to use, as we shall see in later chapters.

Cylindrical Coordinates

We obtain cylindrical coordinates for space by combining polar coordinates in the xy-plane with the usual z-axis. This assigns to every point in space one or more coordinate triples of the form (r, θ, z), as shown in Fig. 10.67.

The values of x, y, r, and θ in cylindrical coordinates are related by the usual equations:

$$x = r\cos\theta, \qquad r^2 = x^2 + y^2, \qquad y = r\sin\theta, \qquad \tan\theta = y/x. \tag{1}$$



10.67 The cylindrical coordinates of a point in space are r, θ, and z.

We shall use cylindrical coordinates to study planetary motion in Section 11.5.

In cylindrical coordinates, the equation $r = a$ describes not just a circle in the xy-plane but an entire cylinder about the z-axis (Fig. 10.68). The z-axis itself is given by the equation $r = 0$. The equation $\theta = \theta_0$ describes the plane that contains the z-axis and makes an angle of θ_0 radians with the positive x-axis.

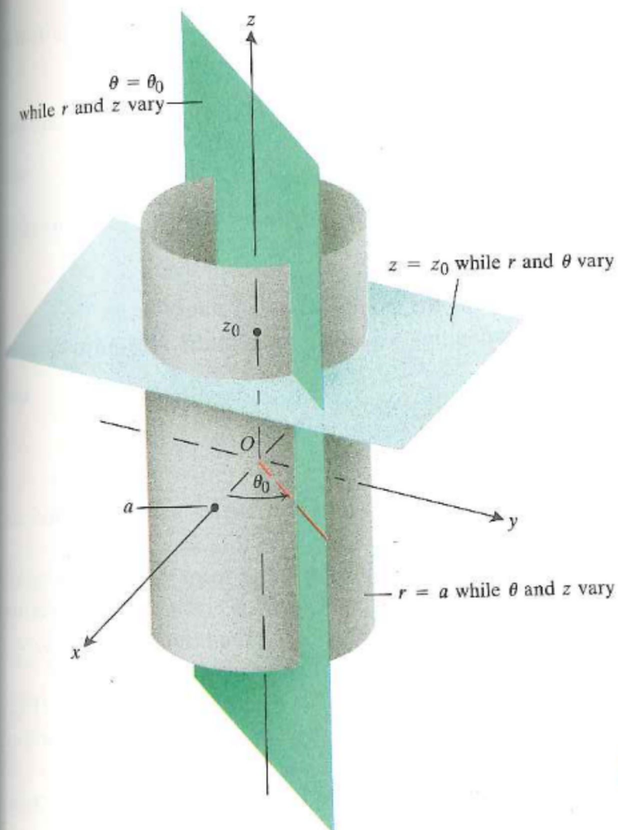
Example 1 Describe the points in space whose cylindrical coordinates satisfy the equations

$$r = 2, \qquad \theta = \frac{\pi}{4}.$$

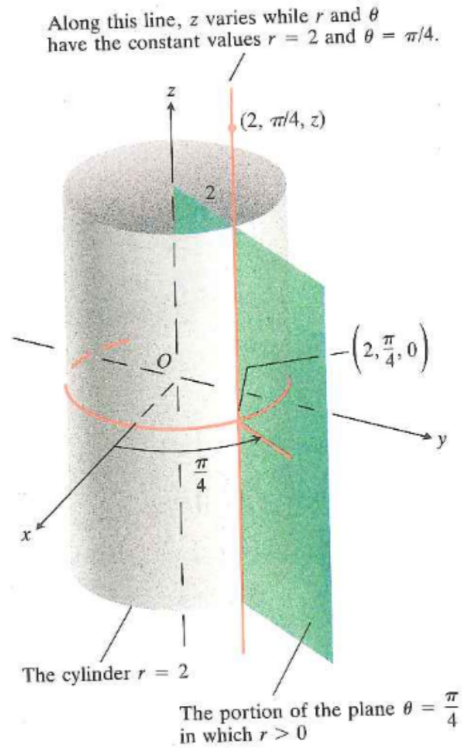
Solution These points make up the line in which the cylinder $r = 2$ cuts the portion of the plane $\theta = \pi/4$ in which r is positive (Fig. 10.69). This is the line through the point $(2, \pi/4, 0)$ parallel to the z-axis.

Example 2 Sketch the surface $r = 1 + \cos\theta$.

Solution The equation involves only r and θ; the coordinate variable z is missing. Therefore, the surface is a cylinder of lines that pass through the cardioid $r = 1 + \cos\theta$ in the $r\theta$-plane and lie parallel to the z-axis. The rules for sketching

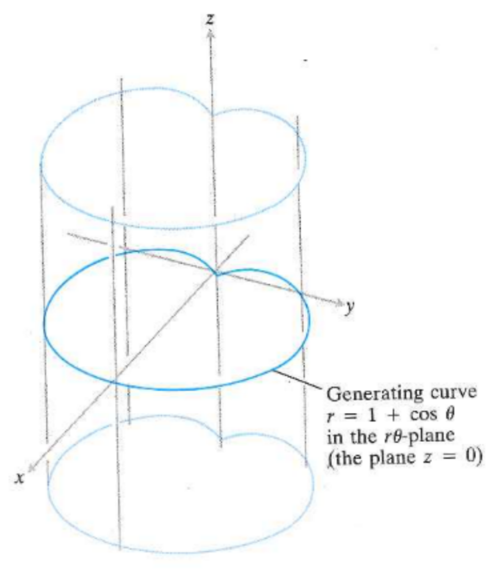


10.68 Planes and cylinders that have constant-coordinate equations in cylindrical coordinates.



10.69 The points whose first two cylindrical coordinates are $r = 2$ and $\theta = \pi/4$ form a line parallel to the z-axis (Example 1).

the cylinder are the same as always: sketch the x-, y-, and z-axes, draw a few perpendicular cross sections, connect the cross sections with parallel lines, and darken the exposed parts (Fig. 10.70).



10.70 The cylindrical-coordinate equation $r = 1 + \cos \theta$ defines a cylinder in space whose cross sections perpendicular to the z-axis are cardioids (Example 2).

Example 3 Find a Cartesian equation for the surface $z = r^2$ and identify the surface.

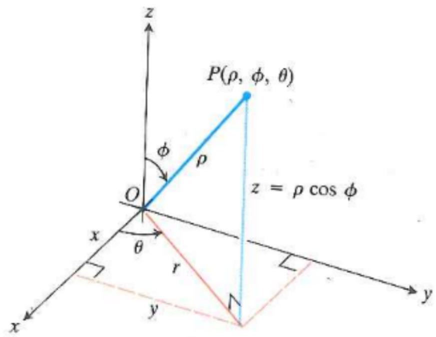
Solution From Eqs. (1) we have $z = r^2 = x^2 + y^2$. The surface is the circular paraboloid $x^2 + y^2 = z$.

Example 4 Find an equation for the circular cylinder $4x^2 + 4y^2 = 9$ in cylindrical coordinates.

Solution The cylinder consists of the points whose distance from the z-axis is $\sqrt{x^2 + y^2} = 3/2$. The corresponding equation in cylindrical coordinates is $r = 3/2$.

Spherical Coordinates

Spherical coordinates locate points in space with two angles and a distance, as shown in Fig. 10.71.



10.71 The spherical coordinates ρ, ϕ, and θ and their relation to x, y, z, and r.

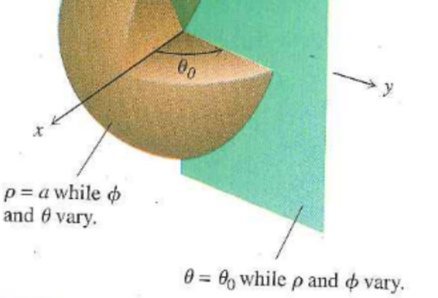
The first coordinate, $\rho = |\overrightarrow{OP}|$, is the point's distance from the origin. Unlike r, the variable ρ is never negative. The second coordinate, ϕ, is the angle the vector $\overrightarrow{OP}$ makes with the positive z-axis. It is required to lie in the interval from 0 to π. The third coordinate is the angle θ from cylindrical coordinates.

The equation $\rho = a$ describes the sphere of radius a centered at the origin (Fig. 10.72). The equation $\phi = \phi_0$ describes a single cone whose vertex lies at the origin and whose axis lies along the z-axis. (We have to broaden our interpretation here to include the xy-plane as the cone $\phi = \pi/2$.) If ϕ_0 is greater than $\pi/2$, the cone $\phi = \phi_0$ opens downward.

A few books give spherical coordinates in the order (ρ, θ, ϕ), with the θ and ϕ reversed. Watch out for this when you read elsewhere.

Selected Equations Relating Cartesian (Rectangular), Cylindrical, and Spherical Coordinates

$$r = \rho \sin \phi, \qquad x = r \cos \theta = \rho \sin \phi \cos \theta,$$
$$z = \rho \cos \phi, \qquad y = r \sin \theta = \rho \sin \phi \sin \theta, \qquad (2)$$
$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$$



10.72 Spheres whose centers are at the origin, single cones at the origin whose axes lie along the z-axis, and half-planes "hinged" along the z-axis have constant-coordinate equations in spherical coordinates.

Example 5 Find a spherical coordinate equation for the sphere

$$x^2 + y^2 + (z - 1)^2 = 1.$$

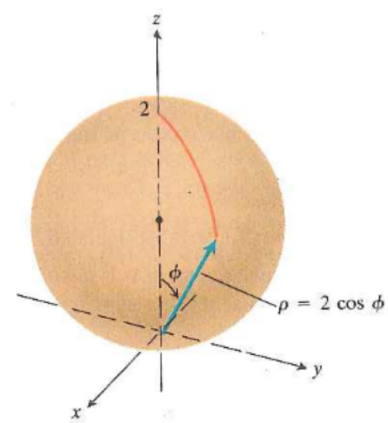
Solution From Eqs. (2) we find that the left side of the equation is

$$\rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta) + \rho^2\cos^2\phi - 2\rho \cos \phi + 1 = \rho^2 - 2\rho \cos \phi + 1.$$

Hence the original equation transforms into

$$\rho^2 - 2\rho \cos \phi + 1 = 1,$$
$$\rho^2 = 2\rho \cos \phi,$$
$$\rho = 2 \cos \phi.$$

See Fig. 10.73.



10.73 The sphere $\rho = 2\cos\phi$ in Example 5. Notice that the equation restricts ϕ to lie in the interval $0 \le \phi \le \pi/2$ because the spherical coordinate ρ is not allowed to be negative.

EXERCISES 10.8

The following table gives the coordinates of specific points in space in one of three coordinate systems. In Exercises 1–10, find coordinates for each point in the other two systems. There may be more than one right answer because points in cylindrical and spherical coordinates may have more than one coordinate triple.

	Rectangular (x, y, z)	Cylindrical (r, θ, z)	Spherical (ρ, ϕ, θ)
1.	$(0, 0, 0)$		
2.	$(1, 0, 0)$		
3.	$(0, 1, 0)$		
4.	$(0, 0, 1)$		
5.		$(1, 0, 0)$	
6.		$(\sqrt{2}, 0, 1)$	
7.		$(1, \pi/2, 1)$	
8.			$(\sqrt{3}, \pi/3, -\pi/2)$
9.			$(2\sqrt{2}, \pi/2, 3\pi/2)$
10.			$(\sqrt{2}, \pi, 3\pi/2)$

In Exercises 11–26, translate the equations from the given coordinate system (rectangular, cylindrical, spherical) into equations in the other two systems. Also, identify the set of points defined by the equation.

11. $r = 0$
12. $x^2 + y^2 = 5$
13. $z = 0$
14. $z = -2$
15. $\rho \cos\phi = 3$
16. $\sqrt{x^2 + y^2} = z$
17. $\rho \sin\phi \cos\theta = 0$
18. $\tan^2\phi = 1$
19. $x^2 + y^2 + z^2 = 4$
20. $\rho = 6\cos\phi$
21. $z = r^2 \cos 2\theta$
22. $z^2 - r^2 = 1$
23. $r = \csc\theta$
24. $z = x^2 + y^2$
25. $3\tan^2\phi = 1$
26. $\rho^2 \cos 2\phi = -1$

In Exercises 27–34, describe the sets of points in space whose cylindrical coordinates satisfy the given equations or pairs of equations. Sketch.

27. $r = 4$
28. $r^2 + z^2 = 1$
29. $r = 1 - \cos\theta$
30. $r = 2\cos\theta$
31. $r = 2, \quad z = 3$
32. $\theta = \pi/6, \quad z = r$
33. $r = 3, \quad z = \theta/2$
34. $r^2 = \cos 2\theta$

35. Find the rectangular coordinates of the center of the sphere

$$r^2 + z^2 = 4r\cos\theta + 6r\sin\theta + 2z.$$

36. What symmetry will you find in a surface whose spherical coordinate equation has the form $\rho = f(\phi)$ (independent of θ)?

In Exercises 37–44, describe the sets of points in space whose spherical coordinates satisfy the given equations or pairs of equations. Sketch.

37. $\phi = \pi/6$
38. $\rho = 6, \quad \phi = \pi/6$
39. $\rho = 5, \quad \theta = \pi/4$
40. $\theta = \pi/4, \quad \phi = \pi/4$
41. $\rho = \cos\phi$
42. $\rho = 1 - \cos\phi$
43. $\rho = \sin\phi$
44. $\theta = \pi/2, \quad \rho = 4\sin\phi$