1. The Tensor Product

Tensor products provide a most "natural" method of combining two modules. They may be thought of as the simplest way to combine modules in a meaningful fashion. As we will see, polynomial rings are combined as one might hope, so that $R[x] \otimes_R R[y] \cong R[x,\, y]$. We will obtain a theoretical foundation from which we may "change the ring" over which a module is defined (it is often the case, for instance, that we wish to consider a real-valued vector space as having a complex extension).

We will be working in the category of R-Mods, and we will consider abelian groups as Z-modules. For R-Mods M and N, the hope is that our "natural" combination $M \otimes N$ is functorial in M and N (from R-Mods to R-Mods).

Definition 1. *Let R be a commutative ring with unit, and let M and N be R-modules. Then, the* ***tensor product*** *$M \otimes_R N$ of M and N is an R-module equipped with a map $M \times N \xrightarrow{\otimes} M \otimes_R N$ that is linear (over R) in both M and N (i.e., a bilinear map). The defining property (up to isomorphism) of this tensor product is that for any R-module P and morphism $f : M \times N \to P$, there exists a unique morphism $\varphi : M \otimes_R N \to P$ such that $f = \varphi \circ \otimes$.*

In general, we will refer to the module of one element (over a given ring) as 0.

Example 1. *Consider the $\mathbb{Z}$-modules $\mathbb{Z}/2$ and $\mathbb{Z}/3$. We claim that $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/3 = 0$. Equivalently, any map $f : \mathbb{Z}/2 \times \mathbb{Z}/3 \to M$ to a $\mathbb{Z}$-module M must be the zero map. One may see this by taking any such f and considering for any $x \in \mathbb{Z}/2$, $y \in \mathbb{Z}/3$,*

$$f(x,\, y) = 3f(x,\, y) - 2f(x,\, y) = f(x,\, 3y) - f(2x,\, y) = f(x,\, 0) - f(0,\, y) = 0.$$

From this example, we begin to understand how to use this object: it is defined by its maps to (or from) other objects, so we must consider those to determine the shape of the module. In fact, in general, we may use similar reasoning to see:

Example 2. *For fixed $a, b \in \mathbb{Z}$, consider the $\mathbb{Z}$-modules $\mathbb{Z}/a$ and $\mathbb{Z}/b$. Then,*

$$\mathbb{Z}/a \otimes_{\mathbb{Z}} \mathbb{Z}/b = \mathbb{Z}/gcd(a,\, b).$$

Even more generally, we may see that:

Theorem 1. *Let R be a commutative ring with unit, and let I and J be ideals of R. Then, considering R/I and R/J as R-modules, we have*

$$R/I \otimes_R R/J = R/(I + J).$$

Example 3. *Consider $R[x]$ as a R-module. Then we have $R[x] \otimes_R R[x] = R[x_1,\, x_2]$, and in general $R[x]^{\otimes k} = R[x_1, \ldots,\, x_k]$, where $R[x]^{\otimes k}$ indicates the k^{th} tensor product of $R[x]$ with itself.*

It is worth noting that by considering the set $\mathrm{Bil}(M,\, N,\, P)$ of bilinear maps from $M \times N$ to P, we have the bijections

$$\mathrm{Bil}(M,\, N,\, P) \cong \mathrm{Hom}_R(M,\, \mathrm{Hom}_R(N,\, P)) \cong \mathrm{Hom}_R(M \otimes_R N, P).$$

This implies that the tensor product is left adjoint to Hom.

Example 4. *Prove the following: $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}$, and $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.*

2. The Construction

A **bilinear** map $T : M \times N \to P$ is a map such that for any $m \in M$, the induced map $T(m, \bullet) : N \to P$ (that is, $f : N \to P$ sending $f(n) = T(m, n)$) is linear and for any $n \in N$, the induced map $T(\bullet, n) : M \to P$ is linear. Note in particular that the map T will not be R-linear in general $-$ in fact, it will be R-linear iff it is the zero map.

Equivalently, we may define a bilinear map $T : M \times N \to P$ to be a map such that for all $m_1, m_2 \in M$, $n_1, n_2 \in N$, and $r \in R$, we have

- $T(m_1 + m_2, n_1) = T(m_1, n_1) + T(m_2, n_1)$
- $T(m_1, n_1 + n_2) = T(m_1, n_1) + T(m_1, n_2)$
- $T(rm_1, n_1) = T(m_1, rn_1) = rT(m_1, n_1)$.

Our previous discussion of tensor products is incomplete; while we may have deduced certain properties of the tensor product in special cases, we have no result stating that the tensor product actually *exists* in general. The following is an explicit construction of a module satisfying the properties of the tensor product.

Definition 2. *Let R be a commutative ring with unit, and let M and N be R-modules. The* ***tensor product*** *$M \otimes_R N$ of M and N is a quotient of the free*

$$F_R(M \times N) := \bigoplus_{(m,n) \in M \times N} R\delta_{(m,n)} \cong R^{M \times N}.$$

Let Δ be the submodule spanned by the elements $\delta_{(m+m',n)} - \delta_{(m,n)} - \delta_{(m,n)}$, $\delta_{(m,n+n')} - \delta_{(m,n)} - \delta_{(m,n')}$, $\delta_{(rm,n)} - r\delta_{(m,n)}$, and $\delta_{(m,rn)} - r\delta_{(m,n)}$, where m and m' vary over elements of M, n and n' vary over elements of N, and r varies over elements of R. Then we define the tensor product $M \otimes_R N := F_R(M \times N)/\Delta$.

One may verify that this definition satisfies the properties outlined in the previous definition and deduce that the tensor product is indeed well-defined. But the details of this correspondence are not very revealing, and as far as we are concerned, this definition is useful only in that it confirms what we previously guessed; the best way to think about the tensor product is in terms of the bilinear maps out of $M \times N$.

It would perhaps be important to note a few properties of the tensor product, especially as it relates to the direct sum.

- (Commutativity) $M \otimes N \cong N \otimes N$.
- (Associativity) $(M \otimes N) \otimes P \cong M \otimes (N \otimes P)$.
- (Distributivity) $M \otimes (N \oplus P) \cong (M \otimes N) \oplus (N \otimes P)$.

Note in particular that associativity justifies the definition of $M^{\otimes k}$ as the tensor product of M with itself k times (otherwise, the order of applying tensor product is ill-defined).