1. Topology

Today, we are going to talk about point-set topology. Point-set topology describes most structures using the concept of continuity, which makes it a general concept with many applications, from measure theory to even abstract algebra. First, let's begin by defining what we mean when we say "topology".

Definition 1. *Given a set X, a topology $\mathcal{T} \subset \mathcal{P}(X)$ is a collection of subsets of X satisfying*
(1) $\emptyset$ and X are in $\mathcal{T}$.
(2) Given an arbitrary $\mathcal{I} \subset \mathcal{T}$,

$$\bigcup_{S \in \mathcal{I}} S \in \mathcal{T}.$$

(3) Given a finite $\mathcal{S} \subset \mathcal{T}$,

$$\bigcap_{S \in \mathcal{S}} S \in \mathcal{T}.$$

That is, the topology is closed under arbitrary unions and finite intersections. These sets in $\mathcal{T}$ are referred to as the **open** sets of our topological space. A set $Y \subset X$ is called **closed** if its complement $X \setminus Y$ is an element of $\mathcal{T}$.

Equivalently, one may define a topological space by specifying a collection of closed sets. In fact, there are three other fundamental ways to define a topology on a space - one may define an **interior** operation on subsets of the space. Intuitively, the interior is the largest open set contained in the subset. Similarly, one may talk about a **closure** operation as the smallest closed set containing the subset. Finally, another method to create a topology is to define **neighborhoods** of points $p \in X$.

1.1. Examples.

- Our first example is the discrete topology. In this topology, all sets are open, and thus all sets are closed.
- Another example is the trivial topology, where the only sets that are open or closed are $\emptyset$ and X.
- The standard topology on $\mathbb{R}$ is generated by the open intervals of $\mathbb{R}$. Similarly, we can generate a topology on $\mathbb{R}^n$ with the open balls.
- In general, any metric space has a natural topology with respect to the given metric - the open sets are those generated by the open balls given by the metric.
- Define the spectrum of a ring R to be the set of all prime ideals of R. Then there is a topology, called the Zariski topology, on R, where the closed sets are given by sets of the form $V(I)$, which is the set of all prime ideals containing some given ideal I. Note that $V(I)$ is nonempty if I is a proper ideal.

We say that a topology $\mathcal{T}_1$ is **finer** than $\mathcal{T}_2$ if $\mathcal{T}_1 \supset \mathcal{T}_2$. Then, $\mathcal{T}_2$ is **coarser** than $\mathcal{T}_1$. The discrete topology is the finest topology on any set, and the trivial topology is the coarsest.

2. Continuity

Let $f : X \to Y$ be a mapping between topological spaces. We say that f is continuous at point $p \in X$ if for any neighborhood N of $f(p)$, there exists a neighborhood M of p such that $f(M) \subset N$. This is equivalent to saying that the pre-image of any neighborhood of $f(p)$ is a neighborhood of p. We say f is **continuous** if f is continuous at every point $p \in X$.

Theorem 1. *The following are equivalent:*

- *f is continuous.*
- *The preimage of any open set is open.*
- *The preimage of any closed set is closed.*
- *For any $E \subset X$, we have $f(\overline{E}) \subset \overline{f(E)}$.*

A **homeomorphism** is a map between topologies whose inverse is also continuous.

3. Convergence and Separability Criteria

Let x_n be a sequence in X. We say that x_n **converges** to a point $p \in X$ if for every neighborhood U_p, we can pick N such that when $n > N$, $x_n \in U$.

3.1. Examples.

- In the familiar Euclidean topology of $\mathbb{R}^n$, this notion of convergence agrees with the usual notion of convergence.
- If a set has the trivial topology, every sequence converges to every point in the set. This example shows that topological convergence need not be unique.
- In a set equipped with the discrete topology, the only convergent sequences are those that become eventually constant.

Note that a closed subset of a topological space must contain the limits of all its convergent sequences. However, the converse is not necessarily true. We tie together this notion of convergence with our earlier discussion of continuity with the following theorem:

Theorem 2. *Let $f : X \to Y$ be a continuous function between two topological spaces. If $x_i \in X$ is a convergent sequence with $\lim x_i = p$ then $f(x_i) \in Y$ is also a convergent sequence with $\lim f(x_i) = f(p)$.*

The separability properties are a series of criteria a space may satisfy, relating to how well differentiated the points are. For example, the points inside the trivial topology are indistinguishable with respect to the topology.

Definition 2. *X is a T_0 space if for any points $p \neq q \in X$, there exists an open set $U \subset X$ which contains only one of those points.*

Definition 3. *X is T_1 if for any points $p \neq q \in X$, there exists an open set $U_p \subset X$ which contains p but not q, and an open set $U_q \in X$ which contains q but not p.*

Note that T_1 is a stricter criteria than T_0. X is a T_1 space if all singleton sets in X are closed.

Definition 4. *X is T_2 if for any points $p \neq q \in X$ there exist open sets $U_p, U_q \subset X$ such that $p \in U_p$, $q \in U_q$, and $U_p \cap U_q = \emptyset$.*

T_2 spaces are also known as **Hausdorff** or **separable** spaces. If a space is Hausdorff, we can know that every sequence in the space converges to at most one point.

Definition 5. *A T_3 space is a T_1 space satisfying the property that for any point $p \in X$ and a closed subset $p \notin Q \subset X$, there exist open sets $U_p, U_q \subset X$ such that $p \in U_p$, $Q \subset U_q$, and $U_p \cap U_q = \emptyset$.*

Definition 6. *A T_4 space is a T_1 space satisfying the property that for any disjoint closed subsets $P, Qsubset X$, there exist open sets $U_p, U_q \subset X$ such that $P \subset U_p$, $Q \subset U_q$, and $U_p \cap U_q = \emptyset$.*

4. Products and Quotients

Definition 7. *The product of a family of topological spaces $\{(X_i, T_i)\}_{i \in I}$ is defined as the product ΠX_i equipped with the coarsest topology for which the canonical projections are continuous.*

Definition 8. *Let (X, T) be a topological space and $\sim$ be an equivalence relation on X. If $\pi : X \to X/\sim$ denotes the canonical quotient map, then $\pi(T)$ is called the quotient topology. It is the finest topology on $X/\sim$ for which the canonical quotient map is continuous.*