1. Dual Objects

Duality is an important general theme that has manifestations in almost every area of mathematics. Many mathematical dualities between objects of two types correspond to pairings, bilinear functions from an object of one type and another object of the second type to some family of scalars. In mathematics, a duality, generally speaking, translates concepts, theorems or mathematical structures into other concepts, theorems or structures, in a one-to-one fashion, often (but not always) by means of an involution operation: if the dual of A is B, then the dual of B is A.

2. Examples

In previous weeks, we have encountered dual vector spaces. You were also introduced to dual matroids on last week's homework. Here are some examples of other dual spaces:

- a dual pair or dual system is a pair of vector spaces with an associated bilinear form. A vector space V together with its algebraic dual V^* and the bilinear form defined as $< x, f >= f(x)$
- In geometry, the platonic solids have a duality between each other. The cube and octahedron are dual, while the dodecahedron and icosahedron are dual. The tetrahedron is self-dual.
- More generally, dual graphs are created by letting the faces of the original graph be vertices of the dual graph, while adding edges between faces that were adjacent in the original graph. Just as graphs represent matroids, this duality operation is analogous to the duality of matroids; in fact, the dual graph's matroid representation will be the dual matroid.
- In projective geometry, there is a duality between points and lines in the projective plane.
- There is a standard duality in Galois theory (fundamental theorem of Galois theory) between field extensions and subgroups of the Galois group: if one has a field extension $F \subset K$, we have an isomorphism sending $Gal(K/L)$ to $F \subset L \subset K$.
- A group of dualities can be described by endowing, for any mathematical object X, the set of morphisms $Hom(X, D)$ into some fixed object D, for example when X is a vector space and D is the underlying field. In general, this yields a true duality only for specific choices of D, in which case $X^* = Hom(X, D)$ is referred to as the dual of X. In some cases, $X^{**} = (X^*)^*$ is isomorphic to X.
- We will see more examples of duality when we cover category theory.

3. Universal Properties

Broadly speaking, an object o endowed with some additional structure S has a corresponding universal property if any other (similar) structure S' imposed upon o can be related back to the original structure S in some way. That is, S is the most "naturally arising" structure on o (abiding by certain restraints).

Often times, this connects to some concept of minimality (in the language of partial orders) or "factoring through" (in the more appropriate language of function composition).

Example 1. *Recall our definition of union of a set S:*

$\bigcup S :=$ *the smallest (by inclusion) set containing each element of S as a subset.*

Then, we may describe a class of sets T satisfying the condition that T contains each element of S as a subset. For any such T, we have a correspondence $T \subset \cup S$.

Example 2. *Given two groups G, H and a group homomorphism $f : G \to H$, let K be a normal subgroup of G such that $K \subseteq \ker f$, and let $\varphi : G \to G/K$ be the natural map identifying an element $g \in G$ with its coset $(g) \in G/K$.*

Then there exists a unique homomorphism $h : G/K \to H$ such that $f = h \circ \varphi$.

Definition 1. *Let G be a group. Then, the commutator subgroup of G is a normal subgroup $\{ghg^{-1}h^{-1} \mid g, h \in G\}$. The* ***abelianization*** *of G is defined to be the quotient group of G by the commutator subgroup, i.e.*

$$G^{ab} := G/\{ghg^{-1}h^{-1} \mid g, h \in G\}.$$

Example 3. *Given a group G, an abelian group A, and a group homomorphism $f : G \to H$, let $\varphi : G \to G^{ab}$ denote the natural map to the quotient group.*

Then there exists a unique homomorphism $h : G^{ab} \to A$ such that $f = h \circ \varphi$.

Example 4. *Let V be a vector space over a field $\mathbb{F}$. Given a set X and a function $g : X \to V$, let $\delta : X \to \mathbb{F}\langle X\rangle$ be the map sending $x \in X$ to the value $\delta_x \in \mathbb{F}\langle X\rangle$ (that is, $\delta_x \in Hom(X, V)$) satisfying $\delta_x(y) = 0$ for all $x \neq y$ and $\delta_x(x) = 1$.*

Then there exists a unique homomorphism $\widehat{g} : \mathbb{F}\langle X\rangle \to V$ such that $g = \widehat{g} \circ \delta$.