Capstone Course

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1 Introduction

- Website

  `https://math.berkeley.edu/~shiyu/`

- Office Hours tentatively 2-4pm Thursday, 938 Evans
- Recommended text:
  Thomas Garrity All the Mathematics You Missed: But Need to Know for Graduate School
- Self-contained course!
- Today, we want to show you how all of our mathematical concepts are built up from very abstract and simple formulations. To do mathematics well, whenever you come across a certain (type of) object, there are two questions you should ask: (1) what are the defining characteristics of this object and (2) what are some important examples of this object.

2 Sets

A set A is a collection of elements. That is, for any object o, we can say either $o \in A$ or $o \notin A$.

- Recall $A \subseteq B$ means $x \in A \Rightarrow x \in B$.
- Recall $\sqcup$ and $\times$ are disjoint union and Cartesian product.
- A relation R defined on elements of a set S is a subset of $S \times S$.
- An equivalence relation is symmetric, reflexive, and transitive.
- A function f from A to B is a subset of $A \times B$ such that for each $a \in A$, there is precisely one b such that $(a, b) \in f$. By convention, we denote $f(a) = b$.
- Recall the powerset, union, and intersection operators.

2.1 Review of basic definitions

Recall that:

- A **set** is a collection of elements.
- A **group** G is a set with an operation on its elements, $*$, that satisfies the following properties:
  - $*$ is associative.
  - There is an element e for which $e*g = g$ and $g*e = g$ for any element $g \in G$.
  - For any $g \in G$ there exists a (unique) element g^{-1} for which $g*g^{-1} = e$ and $g^{-1} * g = e$.
  - An **abelian group** is a group where $*$ is commutative.
- A ring is a set R with two operations $+$ and $*$, so that:
  - R is an abelian group under $+$.
  - $*$ is associative.
  - The operations satisfy the distributive property $a*(b+c) = a*b+a*c$.
  - Most rings we encounter will have a multiplicative identity, known as 1. Some authors take this as a requirement for a ring. Note that the set of even integers is a ring without an identity element, sometimes referred to as a "rng". ⌣̈
- Some other properties a ring may have are:
  - A ring R where $a * b = b * a$ for all $a, b \in R$ is called a **commutative ring**.
  - A nonzero ring R where $a * b = 0$ implies that either a or b is zero is called an **domain**. If it is commutative, it is called an **integral domain**.
  - A **unit** is an element of a ring invertible with respect to $*$. If every element is invertible, R is a **division ring**.
  - A commutative division ring is known as a **field**.
    - Note that an example of a division ring that is not a field is the quarternions.
- A **partially ordered set**, or **poset**, is a set with a relation $\leq$ satisfying the properties of reflexivity, anticommutativity, and transitivity. Note that not all elements in a poset need to be comparable; if they are, then we have a **totally ordered set**.

2.2 Examples of posets

- Consider $[n] = \{1, 2, ..., n\}$. The set $2^{[n]}$ can be made into a poset B_n, the Boolean algebra, by ordering by inclusion.
- Young diagrams...

3 Zorn's Lemon

Applications:

- Existence of basis
- Every nonzero ring has a maximal ideal
- Every field has an algebraic closure
- Tychonoff's Theorem
- In poset, every totally ordered subset is contained in maximal totally ordered subset
- Every infinite set is equivalent to its square
- Well-ordering principle

4 Vector Spaces

- Recall that an **left R-module** over a ring R with unit consists of an abelian group M under $+$, and an operation $R \times M \to M$, known as **scalar multiplication**, for which:
  - $r(x + y) = rx + ry$
  - $(r + s)x = rx + sx$
  - $(rs)x = r(sx)$
  - $1_R x = x$.

  A right R-module is defined similarly.
- Each vector space has a basis because Zorn's Lemma..