1. Categories

In the broadest sense, a category is a concept used to formalize our notion of mathematical objects and their intrinsic structures.

Definition: A **category** $\mathcal{C}$ consists of the following data:

(1) A class of **objects** $\mathrm{ob}(\mathcal{C})$, usually denoted just by $\mathcal{C}$.
(2) For any objects $A, B \in \mathcal{C}$, a set of **morphisms** $\mathrm{Hom}^{\mathcal{C}}(A, B)$.
(3) For any objects $A, B, C \in \mathcal{C}$, a **composition** map $\mathrm{Hom}^{\mathcal{C}}(B, C) \times \mathrm{Hom}^{\mathcal{C}}(A, B) \to \mathrm{Hom}^{\mathcal{C}}(A, C)$ denoted $(f, g) \mapsto f \circ g$.

The composition of morphisms is such that

(a) For any object $A \in \mathcal{C}$, $\mathrm{Hom}^{\mathcal{C}}(A, A)$ contains a distinguished element id_A such that for all $f \in \mathrm{Hom}^{\mathcal{C}}(A, A)$, $f \circ \mathrm{id}_A = f = \mathrm{id}_A \circ f$.
(b) For any objects $A, B, C, D \in \mathcal{C}$ and morphisms $f \in \mathrm{Hom}^{\mathcal{C}}(C, D)$, $g \in \mathrm{Hom}^{\mathcal{C}}(B, C)$, $h \in \mathrm{Hom}^{\mathcal{C}}(A, B)$, there is an associative rule $(f \circ g) \circ h = f \circ (g \circ h)$.

We call a category $\mathcal{C}$ **small** if $\mathrm{ob}(\mathcal{C})$ is actually a set.

Note that not all categories are small; we may consider, for instance, consider **Set**, where the class of objects is all sets and the morphisms are functions from one set to another.

2. Functors

Definition: A (covariant) **functor** $F : \mathcal{C} \to \mathcal{D}$ from category $\mathcal{C}$ to category $\mathcal{D}$ consists of the following data:

(1) For each $A \in \mathcal{C}$, there is an object $F(A) \in \mathcal{D}$.
(2) For each morphism $f : A \to B$ in $\mathcal{C}$, there is a morphism $F(f) : F(A) \to F(B)$.

The mapping of morphisms is such that

(a) For any object $A \in \mathcal{C}$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.
(b) For any morphisms $f : B \to C$, composition is such that $g : A \to B$, $F(f \circ_{\mathcal{C}} g) = F(f) \circ_{\mathcal{D}} F(g)$.

Note that we may now consider, for instance, the category **Cat** of all small categories, the morphisms between which are functors.

3. Isomorphisms

Given a category $\mathcal{C}$ and morphism $f : A \to B$ in $\mathcal{C}$, then we say that f is an **isomorphism** iff there exists some $g : B \to A$ in $\mathcal{C}$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. In such a case, we say A and B are isomorphic or, simply, $A \cong B$.

Definition: Given functors $F, G : \mathcal{C} \to \mathcal{D}$, a **natural transformation** $\eta : F \to G$ consists of the following data:

(1) For each object $A \in \mathcal{C}$, there is a morphism $\eta(A) : F(A) \to G(A)$ in $\mathcal{D}$.
(2) For each morphism $f : A \to B$ in $\mathcal{C}$, there is an equivalence

$$G(f) \circ \eta(A) = \eta(B) \circ F(f).$$

We may then consider, for small category $\mathcal{C}$ and category $\mathcal{D}$, the category $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ in which the objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and the morphisms are natural transformations. This provides us with an immediate concept of **isomorphism of functors**.

4. Representable Functors

Given a category $\mathcal{C}$ and functor $F : \mathcal{C} \to \mathbf{Set}$, we say that F is **representable** iff there exists some $A \in \mathcal{C}$ such that F is isomorphic to $\mathrm{Hom}^{\mathcal{C}}(A, \bullet)$.

Definition: A functor $G(\bullet) := \mathrm{Hom}^{\mathcal{C}}(A, \bullet)$ is defined so that

(1) For each object $B \in \mathcal{C}$, the corresponding $G(B) = \mathrm{Hom}^{\mathcal{C}}(A, B) \in \mathbf{Set}$.
(2) For each morphism $f : B \to C$ in $\mathcal{C}$, the corresponding $G(f) : \mathrm{Hom}^{\mathcal{C}}(A, B) \to \mathrm{Hom}^{\mathcal{C}}(A, C)$ maps $g \mapsto f \circ g$.

Consider, for instance, the category **Grp** of all groups, the morphisms between which are group homomorphisms. An important functor is the forgetful functor $(\bullet)^\flat : \mathbf{Grp} \to \mathbf{Set}$ taking a group to its underlying set and a group homomorphism to itself (as a function and therefore **Set** morphism).

We can note that the forgetful functor is representable, with $(\bullet)^\flat \cong \mathrm{Hom}^{\mathbf{Grp}}(\mathbb{Z}, \bullet)$.

5. Adjoints

A category $\mathcal{C}$ has a dual, obtained by "reversing" the morphisms.

Definition: The **opposite** category $\mathcal{C}^{\mathrm{op}}$ is defined so that

(1) $\mathrm{ob}(\mathcal{C}) = \mathrm{ob}(\mathcal{C}^{\mathrm{op}})$.
(2) For all $A, B \in \mathcal{C}^{\mathrm{op}}$, $\mathrm{Hom}^{\mathcal{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}^{\mathcal{C}}(B, A)$.
(3) Composition $\circ_{\mathcal{C}^{\mathrm{op}}}$ is defined for any $f : A \to B$ and $g : B \to C$ in $\mathcal{C}^{\mathrm{op}}$ so that $f \circ_{\mathcal{C}^{\mathrm{op}}} g = g \circ_{\mathcal{C}} f$.

Given categories $\mathcal{C}$ and $\mathcal{D}$, let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. We say that F is a left **adjoint** of G (equivalently, F has right adjoint G) in the case that we have the following isomorphisms of functors.

(a) For all $A \in \mathcal{C}$, there is $\mathrm{Hom}^{\mathcal{D}}(F(A), \bullet) \cong \mathrm{Hom}^{\mathcal{C}}(A, G(\bullet))$.
(b) For all $B \in \mathcal{D}$, there is $\mathrm{Hom}^{D}(F(\bullet), B) \cong \mathrm{Hom}^{\mathcal{C}}(\bullet, G(B))$.

Note that while the former functors are from $\mathcal{C}$ to **Set**, the latter functors $\mathrm{Hom}^{D}(F(\bullet), B)$ and $\mathrm{Hom}^{\mathcal{C}}(\bullet, G(B))$ are functors from $\mathcal{C}^{\mathrm{op}}$ to **Set**. (In general, these types of functors are referred to as **contravariant** as a "functor" of $\mathcal{C}$.)

Recall that the forgetful functor from **Grp** to **Set** satisfies $(\bullet)^\flat \cong \mathrm{Hom}^{\mathbf{Grp}}(\mathbb{Z}, \bullet)$. The forgetful functor has a left adjoint known as the free functor, i.e. the functor $F : \mathbf{Set} \to \mathbf{Grp}$ taking a set S to the free group $F(S)$ and set morphisms (functions) to the corresponding group homomorphism (determined by the function valued at the free generators). In other words, for all $S \in \mathbf{Set}$ and $G \in \mathbf{Grp}$,

$$\mathrm{Hom}^{\mathbf{Grp}}(F(S), G) \cong \mathrm{Hom}^{\mathbf{Set}}(S, G^\flat)$$

where the isomorphism is given with the structure of **Set**.

6. A Final Note

One important thing to note is the distinction between isomorphism and equality of objects. We say that two objects are equal if they are defined in the same way and actually are the same object. However, an isomorphism is dependent not just on the objects but on the morphisms between the objects.

We already know what it means for two categories to be isomorphic. But there is a weaker notion of equivalence of categories that also warrants some thought.

Definition: Given categories $\mathcal{C}$ and $\mathcal{D}$, we say that $\mathcal{C}$ is **equivalent** to $\mathcal{D}$ if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \cong \mathrm{id}_{\mathcal{C}}$ and $F \circ G \cong \mathrm{id}_{\mathcal{D}}$.