Problem 1. Compute the Fourier series for the function \( f(x) = e^x \) on the interval \([-\pi, \pi]\).

Solution. (This is [1, Section 10.3, Exercise 15].) We have

\[
 f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)
\]

where \( L = \pi \) and

\[
 a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, dx \quad \text{and} \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x \, dx.
\]

We compute \( a_n \) and \( b_n \). We have

\[
 a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos \frac{0}{\pi} x \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \, dx = \frac{e^\pi - e^{-\pi}}{\pi}.
\]

We have

\[
 \frac{d}{dx}(e^x \cos nx) = e^x \cos nx - ne^x \sin nx
\]

\[
 \frac{d}{dx}(e^x \sin nx) = e^x \sin nx + ne^x \cos nx
\]

so

\[
 (n^2 + 1)e^x \cos nx = \frac{d}{dx}(e^x \cos nx) + n \frac{d}{dx}(e^x \sin nx)
\]

\[
 (n^2 + 1)e^x \sin nx = \frac{d}{dx}(e^x \sin nx) - n \frac{d}{dx}(e^x \cos nx)
\]

thus

\[
 a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos nx \, dx
\]

\[
 = \frac{1}{\pi} \left(\frac{1}{n^2 + 1}\right) \left( e^x \cos nx + ne^x \sin nx \right) \bigg|_{-\pi}^{\pi}
\]

\[
 = \frac{(-1)^n(e^\pi - e^{-\pi})}{\pi(n^2 + 1)}
\]

and

\[
 b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin nx \, dx
\]

\[
 = \frac{1}{\pi} \left(\frac{1}{n^2 + 1}\right) \left( e^x \sin nx - ne^x \cos nx \right) \bigg|_{-\pi}^{\pi}
\]

\[
 = \frac{-n(-1)^n(e^\pi - e^{-\pi})}{\pi(n^2 + 1)}.
\]

Thus

\[
 f(x) \sim \frac{e^\pi - e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty} \left( \frac{(-1)^n(e^\pi - e^{-\pi})}{\pi(n^2 + 1)} \cos nx + \frac{-n(-1)^n(e^\pi - e^{-\pi})}{\pi(n^2 + 1)} \sin nx \right).
\]

(In the textbook, the answer is written in terms of \( \sinh x = \frac{e^x - e^{-x}}{2} \).)

Problem 2. Use the method of separation of variables to find the general form of the solution to the heat equation

\[
 \begin{cases}
 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & t \geq 0, x \in [0, \pi] \\
 \frac{\partial u}{\partial x}(0, t) = \frac{\partial^2 u}{\partial x^2}(\pi, t) = 0 & t \geq 0
\end{cases}
\]
Solution. Observe first that the trivial solution \( u(x,t) = 0 \) is a solution. We find nontrivial solutions. We assume that our solution \( u(x,t) \) is of the form \( u(x,t) = X(x)T(t) \). Then \( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \) implies \( X''(x)T(t) = X(x)T'(t) \). If \( T(t) = 0 \) for all \( t \geq 0 \), then \( u(x,t) = 0 \) is the trivial solution. If \( X(x) = 0 \) for all \( x \in [0, \pi/2] \), then \( u(x,t) = 0 \) is the trivial solution. Suppose there exist \( t_0, x_0 \) such that \( T(t_0) \neq 0 \) and \( X(x_0) \neq 0 \). We have \( X''(x_0)T(t_0) = X(x_0)T'(t_0) \). Set \( \lambda = -\frac{X''(x_0)}{X(x_0)} = -\frac{T'(t_0)}{T(t_0)} \). Then \( X''(x)T(t_0) = X(x)T'(t_0) \) implies

\[
X''(x) + \lambda X(x) = 0 \tag{1}
\]

and \( X''(x_0)T(t) = X(x_0)T'(t) \) implies

\[
T'(t) + \lambda T(t) = 0. \tag{2}
\]

We have \( \frac{\partial u}{\partial x}(0, t_0) = X'(x)T(t_0) = 0 \), so \( X'(0) = 0 \). Also, \( \frac{\partial^2 u}{\partial x^2}(0, t_0) = X''(x)T(t_0) = 0 \) so \( X''(x) = 0 \). We find general solution to (1).

Suppose \( \lambda < 0 \). Then the general solution to (1) is \( X(x) = c_1 e^{\sqrt{-\lambda} x} + c_2 e^{-\sqrt{-\lambda} x} \). We have \( X'(x) = \sqrt{-\lambda} c_1 e^{\sqrt{-\lambda} x} - \sqrt{-\lambda} c_2 e^{-\sqrt{-\lambda} x} \) and \( X''(x) = -\lambda c_1 e^{\sqrt{-\lambda} x} - \lambda c_2 e^{-\sqrt{-\lambda} x} \). We have \( X'(0) = 0 \) implies \( \sqrt{-\lambda} c_1 - \sqrt{-\lambda} c_2 = 0 \) so \( c_1 = c_2 \). Also \( X''(0) = -\lambda c_1 e^{\sqrt{-\lambda} \pi} - \lambda c_2 e^{-\sqrt{-\lambda} \pi} = 0 \) so \( c_1 = c_2 = 0 \). Thus we only have the trivial solution.

Suppose \( \lambda = 0 \). Then the general solution to (1) is \( X(x) = c_1 x + c_2 \). We have \( X'(x) = c_1 \) and \( X''(x) = 0 \). Then \( X'(0) = 0 \) implies \( c_1 = 0 \), but there are no conditions imposed on \( c_2 \), so we have the nontrivial solution \( X(x) = c_2 \). Also (2) implies \( T(t) = d \) for some constant \( d \). Thus we have the nontrivial solution \( u(x,t) = a \) where \( a \) is a constant.

Suppose \( \lambda > 0 \). Then the general solution to (1) is \( X(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x \). Then we have \( X'(x) = -\sqrt{\lambda} c_1 \sin \sqrt{\lambda} x + \sqrt{\lambda} c_2 \cos \sqrt{\lambda} x \) and \( X''(x) = -\lambda c_1 \cos \sqrt{\lambda} x - \lambda c_2 \sin \sqrt{\lambda} x \). Then \( X'(0) = 0 \) implies \( c_2 = 0 \), and \( X''(\pi) = 0 \) implies \( -\lambda c_1 \cos \sqrt{\lambda} \pi = 0 \). If \( c_1 = 0 \), then we have the trivial solution. Suppose \( c_1 \neq 0 \). Then \( \cos \sqrt{\lambda} \pi = 0 \). Thus \( \sqrt{\lambda} \pi = \frac{2n-1}{2} \pi \) for some integer \( n \). Thus \( \lambda = (\frac{2n-1}{2})^2 \). Then (2) implies \( T(t) = b e^{-\frac{2n-1}{2}^2 t} \) where \( b \) is a constant. Thus we have the nontrivial solution \( u(x,t) = X(x)T(t) = c_n (\cos \frac{2n-1}{2} x)e^{-\frac{2n-1}{2}^2 t} \) where \( c_n \) is a constant.

Combining the nontrivial solutions obtained from the cases \( \lambda = 0 \) and \( \lambda > 0 \), we get the general solution

\[
u(x,t) = a + \sum_{n=1}^{\infty} c_n \left( \cos \frac{2n-1}{2} x \right) e^{-\frac{(2n-1)^2}{4} t}.
\]

References