Problem 1. Find a general solution to the differential equation \( y'' + 8y' + 16y = 0 \).

Solution. The associated auxiliary equation is \( X^2 + 8X + 16 = (X + 4)^2 \), whose only root is \(-4\) (with multiplicity 2), so the general solution is \( y(x) = c_1 e^{-4x} + c_2 x e^{-4x} \).

Problem 2. Solve the initial value problem: \( y'' + 2y' + y = 0, y(0) = 1, y'(0) = -3 \).

Solution. The associated auxiliary equation is \( X^2 + 2X + 1 = (X + 1)^2 \), whose only root is \(-1\) (with multiplicity 2), so the general solution is \( y(x) = c_1 e^{-x} + c_2 e^{-x} - c_2 x e^{-x} \) so \( y'(0) = -c_1 + c_2 \).

Solving the system \( c_1 = 1, -c_1 + c_2 = -3 \) gives \( c_1 = 1 \) and \( c_2 = -2 \), so our solution is \( y(x) = e^{-x} - 2e^{-x} \).

Problem 3. Find a general solution to the differential equation \( y'''' - y'' + 2y = 0 \).

Solution. The associated auxiliary equation is \( X^3 - X^2 + 2 = (X + 1)(X^2 - 2X + 2) = (X + 1)((X - 1)^2 + 1^2) \), whose roots are \(-1, 1 \pm i\), so the general solution is \( y(x) = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos x + c_4 \sin x \).

(How to factor \( X^3 - X^2 + 2 \)? Look for factors of the form \( X - a \) where \( a \) is an integer. The Rational Root Theorem says that it suffices to check the integers \( a \) which divide the constant coefficient of \( X^3 - X^2 + 2 \), so it suffices to check only \( a \in \{\pm 1, \pm 2\} \).)

Problem 4. Find a general solution to the differential equation \( y^{(4)} + 13y'' + 36y = 0 \).

Solution. The associated auxiliary equation is \( X^4 + 13X^2 + 36 = (X^2 + 4)(X^2 + 9) \) which has roots \( \pm 2i, \pm 3i \), so the general solution is \( y(x) = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos 3x + c_4 \sin 3x \).

(How to factor \( X^4 + 13X^2 + 36 \)? Treat it as a polynomial whose variable is \( X^2 \). Then it becomes a quadratic in \( X^2 \).)

Problem 5. True/False? Justify your answer. Let \( a \neq 0, b, c, d \) be constants, and let \( y_1, y_2, y_3 \) be solutions to \( ay'''' + by'' + cy' + dy = 0 \) on \( \mathbb{R} \). If \( W[y_1, y_2](x_0) = 0 \) for some \( x_0 \in \mathbb{R} \), then \( W[y_1, y_2, y_3](x_0) = 0 \).

Solution. False. Here is a counterexample. Consider the case \( a = 1, b = -1, c = 0, d = 0 \) so that the differential equation is \( y''' - y'' = 0 \). Two solutions to \( y''' - y'' = 0 \) are \( y_1(x) = x \) and \( y_2(x) = e^x \). We have

\[
W[y_1, y_2](x) = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = (x - 1)e^x
\]

which has a root at \( x_0 = 0 \). Set \( y_3(x) = 1 \). Then

\[
W[y_1, y_2, y_3](x) = \begin{vmatrix} x & e^x & 1 \\ 1 & e^x & 0 \\ 0 & e^x & 0 \end{vmatrix} = e^x
\]

which does not vanish at \( x_0 = 1 \).

Problem 6. Show that \( \{1, \ln t, e^t, \sin t\} \) is linearly independent on \((0, \infty)\).

Solution. It suffices to show that the Wronskian \( W[1, \ln t, e^t, \sin t](t) \) does not vanish for some \( t \). We have

\[
W[1, \ln t, e^t, \sin t](t) = \begin{vmatrix} 1 & \ln t & e^t & \sin t \\ 0 & 1/t & e^t & \cos t \\ 0 & -1/t^2 & e^t & -\sin t \\ 0 & 2/t^3 & e^t & -\cos t \end{vmatrix} = \begin{vmatrix} 1/t & e^t & \cos t \\ 1/t^2 & e^t & -\sin t \\ 2/t^3 & e^t & -\cos t \end{vmatrix}
\]

To simplify computation (i.e. checking that the Wronskian does not vanish), we check values of \( t \) which will create the most zeros. If \( t \) is of the form \( n\pi - \frac{\pi}{2} \) for some integer \( n \), then \( \cos t = 0 \) and \( \sin t = \pm 1 \). Let’s choose \( t = \frac{\pi}{2} \). Then

\[
W[1, \ln t, e^t, \sin t](\frac{\pi}{2}) = \begin{vmatrix} 2/\pi & e^{\pi/2} \\ 16/\pi^3 & e^{\pi/2} \end{vmatrix} = e^{\pi/2}(\frac{2\pi^2 - 16}{\pi^3})
\]

which is nonzero since \( \pi \neq 2\sqrt{2} \).

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1 Notation: \( W[y_1, \ldots, y_n](x) \) is the Wronskian of \( \{y_1, \ldots, y_n\} \).