**Problem 1.** Consider the square matrix

\[
A = \begin{bmatrix}
-1 & -2 & -2 \\
4 & 1 & 2 \\
0 & 2 & 1
\end{bmatrix}.
\]

(a) Calculate its characteristic polynomial and its eigenvalues.

(b) Find the eigenvectors of \(A\).

(c) Diagonalize the matrix \(A\).

**Solution.**

(a) The characteristic polynomial of \(A\) is

\[
p_A(t) = \det(A - t \cdot I_3)
\]

\[
= \det \begin{bmatrix}
-1 - t & -2 & -2 \\
4 & 1 - t & 2 \\
0 & 2 & 1 - t
\end{bmatrix}
\]

\[
= (-1 - t) \det \begin{bmatrix}
1 - t & 2 \\
2 & 1 - t
\end{bmatrix} - 4 \det \begin{bmatrix}
-2 & -2 \\
2 & 1 - t
\end{bmatrix}
\]

\[
= (-1 - t)((1 - t)^2 - 2^2) - 4(-2(1 - t) + 2^2)
\]

\[
= (-1 - t)((1 - t)^2 - 2^2) + 8(1 - t)
\]

\[
= (-1 - t)((t - 1)^2 + 4)
\]

\[
= -(t^3 - t^2 + 3t + 5)
\]

hence the eigenvalues of \(A\) are \(-1, 1 \pm 2i\).

(b) We have

\[
E_{-1} = \text{Nul}(A - (-1)I_3) \quad \text{row reduction } \quad \text{Nul} \left[ \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{array} \right] = \text{Span} \left\{ v_1 = \left[ \begin{array}{c}
0 \\
1 \\
-1
\end{array} \right] \right\}
\]

\[
E_{1+2i} = \text{Nul}(A - (1 + 2i)I_3) \quad \text{row reduction } \quad \text{Nul} \left[ \begin{array}{ccc}
1 & 0 & 1 \\
0 & 1 & -i \\
0 & 0 & 0
\end{array} \right] = \text{Span} \left\{ v_2 = \left[ \begin{array}{c}
-1 \\
i \\
1
\end{array} \right] \right\}
\]

\[
E_{1-2i} = \text{Nul}(A - (1 - 2i)I_3) \quad \text{row reduction } \quad \text{Nul} \left[ \begin{array}{ccc}
1 & 0 & 1 \\
0 & 1 & i \\
0 & 0 & 0
\end{array} \right] = \text{Span} \left\{ v_3 = \left[ \begin{array}{c}
-1 \\
-i \\
1
\end{array} \right] \right\}
\]

(c) Let \(\lambda = 1 - 2i\). An eigenvector corresponding to \(\lambda\) is \(v_3\). We have

\[
\text{Re}v_3 = \left[ \begin{array}{c}
-1 \\
0 \\
1
\end{array} \right] \quad \text{and} \quad \text{Im}v_3 = \left[ \begin{array}{c}
0 \\
-1 \\
0
\end{array} \right].
\]
Set
\[
P = \begin{bmatrix} v_1 & \text{Re} v_3 & \text{Im} v_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}
\]
and
\[
C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 2 & 1 \end{bmatrix}.
\]
Then
\[
P^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}
\]
and \( A = PCP^{-1} \). □

Problem 2. (a) Define an inner product in a vector space \( V \).

(b) For \( p, q \in \mathbb{P}_2(\mathbb{R}) \) (polynomials up to degree 2 in one variable, with real coefficients) define
\[
\langle p, q \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1).
\]
Show that this is an inner product.

(c) Find an orthogonal basis in \( \mathbb{P}_2(\mathbb{R}) \) with respect to the above inner product.

Solution. (a) An inner product on a vector space \( V \) is a function \( V \times V \to \mathbb{R} \) which assigns to each ordered pair \( (u, v) \in V \times V \) a scalar \( \langle u, v \rangle \) satisfying the following properties:
(i) \( \langle u_1 + u_2, v \rangle = \langle u_1, v \rangle + \langle u_2, v \rangle \) for any \( u_1, u_2, v \in V \)
(ii) \( \langle cu, v \rangle = c\langle u, v \rangle \) for any \( u, v \in V \) and \( c \in \mathbb{R} \)
(iii) \( \langle u, v \rangle = \langle v, u \rangle \) for any \( u, v \in V \)
(iv) \( \langle u, u \rangle \geq 0 \) for all \( u \in V \), and \( \langle u, u \rangle = 0 \) if and only if \( u = 0 \)

(b) We check the properties listed in part (a). Let \( p, q, r \) be three elements of \( \mathbb{P}_2(\mathbb{R}) \), say
\[
p(t) = a_2 t^2 + a_1 t + a_0 \\
q(t) = b_2 t^2 + b_1 t + b_0 \\
r(t) = c_2 t^2 + c_1 t + c_0.
\]
(i) We have
\[
\langle p(t) + q(t), r(t) \rangle = \langle p(-1) + q(-1), r(-1) \rangle + \langle p(0) + q(0), r(0) \rangle + \langle p(1) + q(1), r(1) \rangle \\
= \langle p(-1)q(-1) + p(0)q(0) + p(1)q(1), r(-1) \rangle + \langle p(-1)q(-1) + p(0)q(0) + p(1)q(1), r(0) \rangle + \langle p(-1)q(-1) + p(0)q(0) + p(1)q(1), r(1) \rangle \\
= \langle p(t), r(t) \rangle + \langle q(t), r(t) \rangle.
\]

\[\footnote{Answers consisting of an example of a vector space \( V \) and an inner product on \( V \) received little credit, unless the four axioms were checked in part (b).}\]
such that $p$ was a polynomial of degree at most 2. There are polynomials at most 8 points, since this is the trickiest part of the proof. Note that it was crucial that $p(1)$ is not the zero polynomial. Consider $p(t) = (t + 1)(t)(t - 1)$.

(ii) We have
\[
\langle cp(t), q(t) \rangle = (cp(-1))q(-1) + (cp(0))q(0) + (cp(1))q(1)
\]
\[
= c(p(-1)q(-1) + p(0)q(0) + p(1)q(1))
\]
\[
= c(p(t), q(t)).
\]

(iii) We have
\[
\langle p(t), q(t) \rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)
\]
\[
= q(-1)p(-1) + q(0)p(0) + q(1)p(1)
\]
\[
= \langle q(t), p(t) \rangle.
\]

(iv) We have
\[
\langle p(t), p(t) \rangle = p(-1)^2 + p(0)^2 + p(1)^2 \geq 0
\]
since the sums of squares of real numbers is always nonnegative. If $p(t) = 0$ (i.e. $p$ is the zero vector in $P_2(\mathbb{R})$), then $\langle p(t), p(t) \rangle = 0^2 + 0^2 + 0^2 = 0$. Conversely, if $p(t) \in P_2(\mathbb{R})$ is a polynomial such that $\langle p(t), p(t) \rangle = 0$, then we must have $p(-1) = p(0) = p(1) = 0$. Since $p(t)$ is a polynomial of degree at most 2, it has at most 2 roots unless it is the zero polynomial. Since $p$ has three roots (i.e. $-1, 0, 1$), we conclude that $p$ is the zero polynomial.\footnote{Answers consisting of "$(p(t), p(t)) = 0$ implies $p(t) = 0$" without further explanation received at most 8 points, since this is the trickiest part of the proof. Note that it was crucial that $p$ was a polynomial of degree at most 2. There are polynomials $p(t)$ of degree greater than 2 such that $p(-1) = p(0) = p(1) = 0$ even though $p(t)$ is not the zero polynomial. Consider $p(t) = (t + 1)(t)(t - 1)$.}

(c) We choose an arbitrary basis $\{1, t, t^2\}$ of $P_2(\mathbb{R})$ and perform Gram-Schmidt on it to obtain an orthogonal basis. Set
\[
u_1 = 1
\]
\[
u_2 = t - \frac{\langle t, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = t - \frac{\langle t, 1 \rangle}{\langle 1, 1 \rangle} = t - \frac{(-1) \cdot 1 + 0 \cdot 1 + 1 \cdot 1}{1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1} = t
\]
\[
u_3 = t^2 - \frac{\langle t^2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle t^2, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2
\]
\[
= t^2 - \frac{\langle t^2, 1 \rangle}{\langle 1, 1 \rangle} 1 - \frac{\langle t^2, t \rangle}{\langle t, t \rangle} t
\]
\[
= t^2 - \frac{1 \cdot 1 + 0 \cdot 1 + 1 \cdot 1}{1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1} - \frac{1 \cdot (-1) + 0 \cdot 0 + 1 \cdot 1}{(-1) \cdot (-1) + 0 \cdot 0 + 1 \cdot 1} t
\]
\[
= t^2 - \frac{2}{3} t.
\]

Hence an orthogonal basis for $P_2(\mathbb{R})$ (relative to the specified inner product) is $\{1, t, t^2 - \frac{2}{3} \}$.

\[\square\]
Problem 3. Find a least squares solution \( \hat{x} \) for the linear system \( Ax = b \), where

\[
A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ -2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ -2 \\ -2 \end{bmatrix}.
\]

Solution. We know that \( \hat{x} \) is a solution to \( A^T A \hat{x} = A^T b \). We have

\[
A^T A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -1 \\ -1 & 11 \end{bmatrix},
\]

and

\[
A^T b = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ -12 \end{bmatrix}.
\]

Now we can solve for \( \hat{x} \):

\[
\begin{bmatrix} 9 & -1 \\ -1 & 11 \end{bmatrix} \begin{bmatrix} \hat{x} \end{bmatrix} = \begin{bmatrix} 10 \\ -12 \end{bmatrix} \implies \hat{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]

\( \square \)

Problem 4. Mark each statement True or False. Justify your answers.

(a) If \( A^4 = A \) then \(-1\) is not an eigenvalue for \( A \).

(b) An invertible matrix has only nonzero eigenvalues.

Solution. (a) True. Suppose \( A^4 = A \); we prove that \(-1\) is not an eigenvalue for \( A \). Let \( v \) be an eigenvector of \( A \) with eigenvalue \( \lambda \). Then \( Av = \lambda v \). Multiplying by \( A^3 \) on both sides gives \( A^4 v = \lambda^4 v \). But \( A^4 = A \) so we have \( \lambda v = \lambda^4 v \). Then \( \lambda \neq -1 \), since otherwise we would have \( -v = v \), which implies \( v = 0 \), which is a contradiction since eigenvectors are nonzero by definition.

(b) True. Let \( A \) be an invertible \( n \times n \) matrix. Suppose for the sake of contradiction that \( 0 \) is an eigenvalue of \( A \). Then there exists a nonzero vector \( v \in \mathbb{R}^n \) such that \( Av = 0v = 0 \). Then the nullspace of \( A \) is nontrivial (because it contains the nonzero vector \( v \)). This contradicts the assumption that \( A \) is invertible. Hence \( 0 \) is not an eigenvalue of \( A \).

\( \square \)

Problem 5. A \( 3 \times 3 \) symmetric matrix \( A \) has eigenvalues \( \lambda_1 = 0 \), \( \lambda_2 = 1 \), and \( \lambda_3 = 2 \). The first two\(^3\) eigenvectors are

\[
v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} ; \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.
\]

(a) Find the third eigenvector \( v_3 \).\(^4\)

(b) Find the matrix \( A \).

\(^3\)i.e. \( v_1 \) has eigenvalue \( \lambda_1 \) and \( v_2 \) has eigenvalue \( \lambda_2 \)

\(^4\)i.e. find an eigenvector \( v_3 \) corresponding to the eigenvalue \( \lambda_3 \).
Solution. (a) Let $v_3$ be an eigenvector corresponding to $\lambda_3$. Since $A$ is (real) symmetric, we know that eigenvectors corresponding to different eigenvalues are orthogonal. In other words, we know that $v_1 \cdot v_3 = 0$ and $v_2 \cdot v_3 = 0$.

Suppose $v_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Then $v_1 \cdot v_3 = 0$ implies $a + c = 0$. Then $v_2 \cdot v_3 = 0$ implies $a - c = 0$. This implies $a = c = 0$. Since $v_3$ is nonzero, we must have $b \neq 0$. Thus the eigenspace corresponding to $\lambda_3$ is Span $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(b) Combine the vectors $v_1, v_2, v_3$ into a single matrix $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$.

Let $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Then $AP = PD$, since the $i$th column of $P$ is $v_i$, which is an eigenvector of eigenvalue $\lambda_i$, which is the $i$th diagonal entry of $D$. Thus $A = PDP^{-1}$. Using your favorite method, compute $P^{-1} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \end{bmatrix}$. Then

$A = PDP^{-1}$

$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1/2 & 0 & -1/2 \\ 1/2 & 0 & -1/2 \\ 0 & 2 & 0 \end{bmatrix}$.

(Note: it is not true that $P^{-1} = P^T$ if the columns of $P$ are not normal. It is true that $P^TP$ is diagonal (just not the identity).)

□

Problem 6. Prove the following inequalities for vectors in an inner product space $V$:

(a) For any two vectors $u, v$ we have

$\langle u, v \rangle^2 \leq \|u\|^2 \cdot \|v\|^2$

(b) If $v_1, \ldots, v_k$ is an orthonormal set, then for each vector $x$ we have

$\|x\|^2 \geq \langle x, v_1 \rangle^2 + \ldots + \langle x, v_k \rangle^2$.

Solution. (a) See Theorem 16 in Section 6.7.

(b) For any vector $x$ and any subspace $W$ of $V$, we have $\|\text{proj}_W x\| \leq \|x\|$ (the length of the projection of any vector onto any subspace is at most the

Answers stating “$v_3 = [0, 1, 0]^T$” without further justification were given 5 points.
length of the vector itself). Let $W = \text{Span}\{v_1, \ldots, v_k\}$. Then

$$\text{proj}_W x = \frac{\langle x, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \cdots + \frac{\langle x, v_k \rangle}{\langle v_k, v_k \rangle} v_k$$

$$= (x, v_1) v_1 + \cdots + (x, v_k) v_k$$

where the last step follows because each $v_i$ is a unit vector. Then

$$\|x\|^2 \geq \|\text{proj}_W x\|^2$$

$$= (\text{proj}_W x, \text{proj}_W x)$$

$$= \langle (x, v_1) v_1 + \cdots + (x, v_k) v_k, (x, v_1) v_1 + \cdots + (x, v_k) v_k \rangle$$

$$= \sum_{i=1}^k \sum_{j=1}^k \langle (x, v_i) v_i, (x, v_j) v_j \rangle \text{ by bilinearity}$$

$$= \sum_{i=1}^k \sum_{j=1}^k \langle x, v_i \rangle \langle x, v_j \rangle \langle v_i, v_j \rangle$$

$$= \sum_{i=1}^k \langle x, v_i \rangle \langle x, v_i \rangle \langle v_i, v_i \rangle \text{ since } \langle v_i, v_j \rangle = 0 \text{ if } i \neq j$$

$$= \langle x, v_1 \rangle^2 + \cdots + \langle x, v_k \rangle^2 .$$

\square