The invertible matrix theorem (Lay, Section 2.3, Theorem 8) is a really important
theorem, and it’s really hard to learn and remember 12 conditions. It may be
helpful to organize the conditions in the following way.

Let’s fix the following notation: Let $A$ be a $m \times n$ matrix, $v_1, \ldots, v_n$ the columns
of $A$, and $f : \mathbb{R}^n \to \mathbb{R}^m$ the linear function given by $f(x) = Ax$.

1. Lemmas about injectivity and surjectivity

Injectivity of $f$ means “$f$ doesn’t lose any information about the domain $\mathbb{R}^n$”.

**Lemma 1** (Injectivity). The following are equivalent conditions:

1. $Ax = 0$ has only the trivial solution.
2. $v_1, \ldots, v_n$ are linearly independent.
3. $f$ is one-to-one (injective).
4. There is a pivot in every column.
5. There is an $n \times m$ matrix $C$ such that $CA = I_n$.

If the above conditions hold, then $m \geq n$.

Surjectivity of $f$ means “$f$ doesn’t lose any information about the range $\mathbb{R}^m$”.

**Lemma 2** (Surjectivity). The following are equivalent conditions:

1. $Ax = b$ has a solution for all $b \in \mathbb{R}^m$.
2. $\mathbb{R}^m = \text{Span}\{v_1, \ldots, v_n\}$.
3. $f$ is onto (surjective).
4. There is a pivot in every row.
5. There is an $n \times m$ matrix $D$ such that $AD = I_m$.

If the above conditions hold, then $m \leq n$.

Comments 3. The $i$th condition of Lemma 1 is analogous to the $i$th condition
of Lemma 2. Conditions (4-a) and (5-a) are phrased in the language of matrix
equations. Conditions (4-b) and (5-b) are in the language of vector equations.
Conditions (4-c) and (5-c) are in the language of functions. Conditions (4-d) and
(5-d) are about the number of pivots; both say that there are as many pivots
that the matrix can possibly allow (since pivot positions are in different rows and
columns, a $m \times n$ matrix can contain at most $\min\{m, n\}$ pivot positions). Condition
(4-e) and (5-e) relate to the invertibility of matrix $A$. 
Warning about (4-e) and (5-e): if \( m \neq n \), then (4-e) does not necessarily imply (5-e) and vice versa. Consider the case \( m = 2 \) and \( n = 1 \) and \( A = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). Then (4-e) holds; we can let \( C = \begin{bmatrix} 1 & 0 \end{bmatrix} \) (or \( C = \begin{bmatrix} 1 & a \end{bmatrix} \) for any number \( a \)). But (5-e) does not hold, since if \( D = \begin{bmatrix} d_1 & d_2 \end{bmatrix} \) then
\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} d_1 & d_2 \end{bmatrix} = \begin{bmatrix} d_1 & d_2 \\ 0 & 0 \end{bmatrix},
\]
which can never equal \( I_2 \).

2. Statement of the Theorem and Applications

In what follows, we assume that \( m = n \) (the dimension of the domain and range of \( f \) are the same; the number of rows of \( A \) and the number of columns of \( A \) are the same).

**Theorem 4** (Invertible Matrix Theorem). Suppose \( m = n \). The following are equivalent conditions:

1. \( A \) is invertible.
2. \( A \) is row equivalent to \( I_n \).
3. \( A \) has \( n \) pivot positions.
4. \{All the conditions of Lemma 1\}
5. \{All the conditions of Lemma 2\}
6. \( A^T \) is invertible.

**Proposition 5.** A linear map \( f : \mathbb{R}^n \to \mathbb{R}^n \) is injective if and only if it is surjective.