LINEAR DIFFERENTIAL EQUATIONS

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1. Existence and Uniqueness

Theorem 1 (Existence and Uniqueness). [1, NSS, Section 6.1, Theorem 1] suppose \( p_1(x), \ldots, p_n(x) \) and \( g(x) \) are continuous real-valued functions on an interval \((a, b)\) that contains the point \( x_0 \). Then, for any choice of (initial values) \( \gamma_0, \ldots, \gamma_{n-1} \), there exists a unique solution \( y(x) \) on the whole interval \((a, b)\) to the nonhomogeneous differential equation

\[
y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = g(x)
\]

for all \( x \in (a, b) \) and \( y^{(i)}(x_0) = \gamma_i \) for \( i = 0, \ldots, n-1 \).

Corollary 2. Suppose \( p_1(x), \ldots, p_n(x) \) are continuous real-valued functions on an interval \((a, b)\) containing the point \( x_0 \). Let \( V \) be the set of solutions to the homogeneous differential equation

\[
y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = 0.
\]

Then \( V \) is a vector space, and the function \( T : V \to \mathbb{R}^n \) defined by

\[
y(x) \mapsto \begin{bmatrix} y(x_0) \\ y'(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix}
\]

is a linear isomorphism. In particular, \( V \) is an \( n \)-dimensional vector space.

Proof. Check that \( V \) is closed under addition, scalar multiplication, and contains the zero vector (zero function). Check that \( T \) is linear. Surjectivity of \( T \) follows from the existence part of Theorem 1. Injectivity of \( T \) follows from the uniqueness part of Theorem 1. \( \square \)

As was the case for systems of linear equations (see [1, Lay, Section 1.5]), the set of solutions to a homogeneous system (1) is a subspace of the vector space of continuous real-valued functions on \((a, b)\), and the set of solutions to a nonhomogeneous system (2) is a translation of this subspace by a particular solution \( y_p \).

Corollary 3. Suppose \( p_1(x), \ldots, p_n(x) \) are differentiable real-valued functions (for example, constants) on an interval \((a, b)\). Then any solution \( y \) to (2) is smooth (i.e. \( y \) is infinitely differentiable). Furthermore, the values of \( y^{(k)}(x_0) \) for \( k \geq n \) are determined by \( y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0) \).

\[ ^{1}\text{Notation: } y^{(n)} = \frac{d^n}{dx^n}. \]

\[ ^{2}\text{It is implicit that any solution } y(x) \text{ to (2) is } n\text{-times differentiable (i.e. } y', y'', \ldots, y^{(n)} \text{ all exist).} \]
Proof. Rearrange (2) to get
\[ y^{(n)}(x) = -(p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x)) \]
whose RHS is differentiable (since it is obtained as the sum of products of differentiable functions). Thus \( y^{(n)}(x) \) is differentiable. Differentiate inductively to get
\[ y^{(k)}(x) = -(p_1(x)y^{(k-1)}(x) + \cdots + p_n(x)y^{(k-n)}(x)) \]
for any \( k \geq n \).

Note that [1, NSS, Section 4.2, Theorem 1] is a special case of [1, NSS, Section 6.1, Theorem 1] when \( n = 2 \), the function \( g(x) \) is identically zero, and \( (a, b) = (-\infty, \infty) = \mathbb{R} \).

**Proposition 4** (Extension of solutions to a larger interval). Suppose \( p_1(x), \ldots, p_n(x) \) and \( g(x) \) are continuous real-valued functions on \( \mathbb{R} \). Suppose \( -\infty \leq a_2 \leq a_1 < b_1 \leq b_2 \leq \infty \), and suppose \( y : (a_1, b_1) \rightarrow \mathbb{R} \) is a solution to (1). Then there exists a unique function \( \tilde{y} : (a_2, b_2) \rightarrow \mathbb{R} \) satisfying (1) for all \( x \in (a_2, b_2) \) and \( \tilde{y}(x) = y(x) \) for all \( x \in (a_1, b_1) \).

Proof. Choose \( t_0 \in (a_1, b_1) \). By the existence part of Theorem 1 (applied to the case \( a = a_2, b = b_2 \)), there exists a function \( \hat{y} : (a_2, b_2) \rightarrow \mathbb{R} \) which satisfies (1) and has the same initial conditions as \( y \), i.e. \( \hat{y}^{(k)}(t_0) = y^{(k)}(t_0) \) for all \( k = 0, \ldots, n-1 \). Since \( \hat{y} \) and \( y \) have the same initial conditions at \( t_0 \) and satisfy the same differential equation (1), the uniqueness part of Theorem 1 (applied to the case \( a = a_1, b = b_1 \)) shows that \( \hat{y}(t) = y(t) \) for all \( t \in (a_1, b_1) \). Thus such \( \hat{y} \) having the desired properties exists.

To show that such \( \hat{y} \) is unique, suppose that \( \hat{y} : (a_2, b_2) \rightarrow \mathbb{R} \) also satisfies (1) and has the same initial conditions as \( y \). The uniqueness part of Theorem 1 (applied to \( a = a_2, b = b_2 \)) shows that \( \hat{y}(t) = \tilde{y}(t) \) for all \( t \in (a_2, b_2) \), so that \( \hat{y} = \tilde{y} \). \( \square \)

2. The Wronskian

**Definition 5.** Let \( y_1, \ldots, y_n \) be \((n-1)\)-times differentiable functions. The **Wronskian** of \( y_1, \ldots, y_n \) is the function
\[ W[y_1, \ldots, y_n](x) := \det M_{y_1, \ldots, y_n}(x) \]
where
\[ M_{y_1, \ldots, y_n}(x) := \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix} = \begin{bmatrix} T(y_1(x)) & T(y_2(x)) & \cdots & T(y_n(x)) \end{bmatrix} \]
where \( T \) is the linear isomorphism in Corollary 2.

**Proposition 6.** Let \( y_1, \ldots, y_n \) be \((n-1)\)-times differentiable functions on \((a, b)\). Suppose that \( \{y_1, \ldots, y_n\} \) is linearly dependent. Then \( W[y_1, \ldots, y_n](x) = 0 \) for all \( x \in (a, b) \). \( \square \)

\( ^3 \)Note that Definition 5 and Proposition 6 make no reference to the equation (1).
Proof. Let \( c_1, \ldots, c_n \) be scalars, not all zero, such that
\[
c_1 y_1(x) + \cdots + c_n y_n(x) = 0 \tag{3}
\]
for all \( x \in (a, b) \). Differentiating (3) \( k \) times gives
\[
c_1 y_1^{(k)}(x) + \cdots + c_n y_n^{(k)}(x) = 0.
\]
This means
\[
M_{y_1, \ldots, y_n}(x) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{4}
\]
for all \( x \in (a, b) \). Thus \( M_{y_1, \ldots, y_n}(x) \) is not invertible for all \( x \). Thus \( W[y_1, \ldots, y_n](x) = \det M_{y_1, \ldots, y_n}(x) = 0 \) for all \( x \).

Corollary 7. Let \( y_1, \ldots, y_n \) be \((n-1)\)-times differentiable functions on \((a, b)\). Suppose \( W[y_1, \ldots, y_n](x_0) \neq 0 \) for some \( x_0 \in (a, b) \). Then \( \{y_1, \ldots, y_n\} \) is linearly independent on \((a, b)\).

Proof. This is the contrapositive of Proposition 6.

Example 8. The converse to Proposition 6 is false. Consider \( y_1(x) = x^2 \) and \( y_2(x) = x \cdot |x| \). Then \( y_1'(x) = 2x \) and \( y_2'(x) = 2|x| \). Thus
\[
W[y_1, y_2](x) = \det \begin{bmatrix} x^2 & x \cdot |x| \\ 2x & 2|x| \end{bmatrix} = 0
\]
for all \( x \in \mathbb{R} \), but \( x^2 \) and \( x \cdot |x| \) are linearly independent on \( \mathbb{R} \). (They are linearly dependent on \((-\infty, 0)\) and \((0, \infty)\), separately, though.)

Theorem 9 (Wronskian Criterion). [1, NSS, Section 6.1, Theorem 2] Suppose \( p_1(x), \ldots, p_n(x) \) are continuous real-valued functions on an interval \((a, b)\). Suppose \( y_1, \ldots, y_n \) are \( n \) arbitrary solutions to (2). Then the following are equivalent:

(i) \( \{y_1, \ldots, y_n\} \) is linearly independent;

(ii) the Wronskian \( W[y_1, \ldots, y_n](x) \neq 0 \) for all \( x \in (a, b) \).

In other words, the following are equivalent:

(i') \( \{y_1, \ldots, y_n\} \) is linearly dependent;

(ii') the Wronskian \( W[y_1, \ldots, y_n](x_0) = 0 \) for some \( x_0 \in (a, b) \).

Proof. (i') \( \implies \) (ii'): Follows directly from Proposition 6.

(ii') \( \implies \) (i'): Since \( W[y_1, \ldots, y_n](x_0) = 0 \), we can find scalars \( c_1, \ldots, c_n \), not all zero, satisfying (4). This means \( c_1 T(y_1(x)) + \cdots + c_n T(y_n(x)) = 0 \) (where \( T \) is the linear isomorphism defined in Corollary 2). Since \( T \) is injective, we have \( c_1 y_1(x) + \cdots + c_n y_n(x) = 0 \). Thus \( \{y_1(x), \ldots, y_n(x)\} \) is linearly dependent.

\[^4\text{Why couldn’t we do this with } x \text{ and } |x|? \text{ This is because we need a second power of } x \text{ to “smooth out” } |x| \text{ at the origin (i.e. make it differentiable).} \]

\[^5\text{It is important to note that linear independence is conditional on the domain of the functions. For example, } \{t, |t|\} \text{ is linearly dependent when the domain is } (0, \infty) \text{ but linearly independent when the domain is } (-\infty, \infty). \]
Corollary 10. Suppose \( p_1(x), \ldots, p_n(x) \) are continuous real-valued functions on an interval \((a,b)\). Suppose \( y_1, \ldots, y_n \) are arbitrary solutions to (2). Then the Wronskian \( W[y_1, \ldots, y_n](x) \) is either always zero or always nonzero.

Example 11. Theorem 9 requires that you take the Wronskian of \( n \) solutions, where \( n \) is the order of the differential equation. The criterion is no longer true if you compute the Wronskian of less than \( n \) solutions. For example, consider \( y_2(x) = x \) and \( y_3(x) = e^x \), which are both solutions to the 3rd order differential equation \( y''' - y'' = 0 \). We have

\[
W[y_2, y_3](x) = \det \begin{bmatrix} x & e^x \\ 1 & e^x \end{bmatrix} = (x - 1)e^x
\]

which has a root at \( x = 1 \) even though \( y_2, y_3 \) are linearly independent on any interval (which, at first glance, seems to contradict Theorem 9). Only when you throw in the third basis element (of the space of solutions to \( y''' - y'' = 0 \)), for example \( y_1(x) = 1 \), do you get

\[
W[y_1, y_2, y_3](x) = \det \begin{bmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{bmatrix} = e^x
\]

which has no roots (i.e. always nonzero).

3. Linear second-order constant-coefficient differential equations

(This is from [1, NSS, Section 4.2, 4.3].) We consider the vector space \( V \) of solutions \( y(x) \) to the differential equation

\[
ay''' + by' + cy = 0
\]

where \( a \neq 0 \). By Corollary 2, we have that \( V \) is a 2-dimensional vector space. The associated characteristic equation is

\[
ar^2 + br + c = 0
\]

which has discriminant \( b^2 - 4ac \).

(i) If \( b^2 - 4ac > 0 \), then (6) has two distinct real roots, say \( r_1 \) and \( r_2 \). We observe that \( y_1(x) = e^{rx} \) and \( y_2(x) = xe^{rx} \) are two elements of \( V \) (i.e. they are solutions to (5)), and \( \{y_1, y_2\} \) is linearly independent (which can be shown by computing the Wronskian). Since \( \dim V = 2 \), we have \( V = \text{Span}\{y_1, y_2\} \).

(ii) If \( b^2 - 4ac = 0 \), then (6) has one real root of multiplicity 2, say \( r \). We observe that \( y_1(x) = e^{rx} \) and \( y_2(x) = xe^{rx} \) are two elements of \( V \) (i.e. they are solutions to (5)), and \( \{y_1, y_2\} \) is linearly independent (which can be shown by computing the Wronskian). Since \( \dim V = 2 \), we have \( V = \text{Span}\{y_1, y_2\} \).

(iii) If \( b^2 - 4ac < 0 \), then (6) has two complex conjugate roots, say \( r + ci \) and \( r - ci \). We observe that \( y_1(x) = e^{(r+ci)x} = e^{rx}(\cos cx + i \sin cx) \) and \( y_2(x) = e^{(r-ci)x} = e^{rx}(\cos cx - i \sin cx) \) are two elements of \( V \) (i.e. they are solutions to (5)), and \( \{y_1, y_2\} \) is linearly independent (which can be shown by computing the Wronskian). Since \( \dim V = 2 \), we have \( V = \text{Span}\{y_1, y_2\} \).
References