Suppose we are asked to find the integral \( \int_a^b x^4 \, dx \) using the limit definition. We start to compute

\[
\int_a^b x^4 \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left( a + \frac{b-a}{n} i \right)^4 \frac{b-a}{n}
\]

\[
= \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} \left( a^4 + 4a^3 \frac{b-a}{n} i + 6a^2 \left( \frac{b-a}{n} \right)^2 i^2 + 4a \left( \frac{b-a}{n} \right)^3 i^3 + \left( \frac{b-a}{n} \right)^4 i^4 \right)
\]

and we know \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \) and \( \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}, \) but you may not know what the formula for \( \sum_{i=1}^{n} i^k \) is. The purpose of this document is to explain how to find \( \sum_{i=1}^{n} i^k \) for any \( n \) and \( k \). To this end, define

\[
S_{n,k} := \sum_{i=1}^{n} i^k = 1^k + 2^k + \cdots + (n-1)^k + n^k
\]

and suppose that you want to find \( S_{n,k} \) in terms of \( S_{n,0}, S_{n,1}, \ldots, S_{n,k-1} \). We have

\[
S_{n,k+1} = \binom{k+1}{0} 1^{k+1} + \binom{k+1}{1} 2^{k+1} + \cdots + \binom{k+1}{n-1} (n-1)^{k+1} + \binom{k+1}{n} n^{k+1}
\]

\[
\begin{align*}
\binom{k+1}{0} S_{n,0} & = \binom{k+1}{1} 1^1 + \binom{k+1}{k} 2^1 + \cdots + \binom{k+1}{n-1} (n-1)^1 + \binom{k+1}{n} n^1 \\
\binom{k+1}{k} S_{n,k} & = \binom{k+1}{k+1} 1^0 + \binom{k+1}{k+1} 2^0 + \cdots + \binom{k+1}{k+1} (n-1)^0 + \binom{k+1}{k+1} n^0
\end{align*}
\]

by the Binomial Theorem, so adding the columns above gives

\[
\sum_{i=0}^{k+1} \binom{k+1}{i} S_{n,k+1-i} = (1+1)^{k+1} + (2+1)^{k+1} + \cdots + (n-1+1)^{k+1} + (n+1)^{k+1}
\]

\[
= 2^{k+1} + 3^{k+1} + \cdots + n^{k+1} + (n+1)^{k+1} = S_{n+1,k+1} - 1
\]
and we have $S_{n+1,k+1} = S_{n,k+1} + (n + 1)^{k+1}$ so

\[ \sum_{i=0}^{k+1} \binom{k+1}{i} S_{n,k+1-i} = S_{n,k+1} + (n + 1)^{k+1} - 1 \]

and

\[ \sum_{i=1}^{k+1} \binom{k+1}{i} S_{n,k+1-i} = (n + 1)^{k+1} - 1 \]

thus

\[ (k+1)S_{n,k} = (n + 1)^{k+1} - 1 - \sum_{i=2}^{k+1} \binom{k+1}{i} S_{n,k+1-i} \]

and

\[
S_{n,k} = \frac{1}{k+1} \left( (n + 1)^{k+1} - 1 - \sum_{i=2}^{k+1} \binom{k+1}{i} S_{n,k+1-i} \right).
\]

Let me verify the formula for the case $k = 3$:

\[
S_{n,3} = \frac{1}{4} \left( (n + 1)^4 - 1 - \left( 6 \cdot \frac{n(n + 1)(2n + 1)}{6} + 4 \cdot \frac{n(n + 1)}{2} + 1 \cdot n \right) \right)
\]

\[
= \frac{1}{4} \left( (n + 1)^4 - 1 - (n(n + 1)(2n + 1) + 2n(n + 1) + n) \right)
\]

\[
= \frac{n + 1}{4} \left( (n + 1)^3 - (n(2n + 1) + 2n + 1) \right)
\]

\[
= \frac{n + 1}{4} \left( n^3 + n^2 \right)
\]

\[
= \left( \frac{n(n + 1)}{2} \right)^2.
\]