MATH 1A QUIZ 1 SOLUTION

Please write your solutions on a separate sheet of paper. Be sure to write your name and section number at the top of each page.

Problem 1. (15 points)
(i) State the Squeeze Theorem.
(ii) Prove the Squeeze Theorem.
(iii) Use the Squeeze Theorem to find
\[ \lim_{x \to 0} \frac{x^4}{10} \cos \frac{2\pi}{5x} . \]

Solution. (i) If
\[ f(x) \leq g(x) \leq h(x) \] (1)
when \( x \) is near \( a \) (except possibly at \( a \)) and if \( \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \), then \( \lim_{x \to a} g(x) = L \). The meaning of “near \( a \) (except possibly at \( a \))” is: there exists some \( \delta_1 \) such that for all \( x \) satisfying \( 0 < |x - a| < \delta_1 \), we have (1).

(ii) Suppose \( f, g, h \) satisfy (1) when \( 0 < |x - a| < \delta_1 \) for some fixed \( \delta_1 \). Also suppose that \( \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \). Let \( \varepsilon > 0 \). Since \( \lim_{x \to a} f(x) = L \), there exists \( \delta_2 > 0 \) such that if \( 0 < |x - a| < \delta_2 \), then \( |f(x) - L| < \varepsilon \). Also, since \( \lim_{x \to a} h(x) = L \), there exists \( \delta_3 > 0 \) such that if \( 0 < |x - a| < \delta_3 \), then \( |h(x) - L| < \varepsilon \). Put \( \delta = \min\{\delta_1, \delta_2, \delta_3\} \). Suppose \( 0 < |x - a| < \delta \). Then (1), \( L - \varepsilon < f(x) < L + \varepsilon \) and \( L - \varepsilon < h(x) < L + \varepsilon \) hold. Hence \( g(x) \leq h(x) < L + \varepsilon \). Also \( L - \varepsilon < f(x) \leq g(x) \). Thus \( L - \varepsilon < g(x) < L + \varepsilon \), so \( |g(x) - L| < \varepsilon \). Hence \( \lim_{x \to a} g(x) = L \).

(iii) Set \( f(x) = -\frac{x^4}{10} \), \( g(x) = \frac{x^4}{10} \cos \frac{2\pi}{5x} \), and \( h(x) = \frac{x^4}{10} \). Note that
\[ -1 \leq \cos \frac{2\pi}{5x} \leq 1 \] (2)
for all \( x \neq 0 \), so multiplying (2) by \( \frac{x^4}{10} \) implies that (1) holds for all \( x \neq 0 \). (Here we can take \( \delta_1 \) to be any positive real number.) Note that \( \lim_{x \to 0} f(x) = \lim_{x \to 0} h(x) = 0 \). Hence, by the Squeeze Theorem, \( \lim_{x \to 0} g(x) = 0 \).

\( \square \)

Problem 2. (30 points)
(i) State the definition of limit for sequences (i.e. what exactly does \( \lim_{n \to \infty} f(n) = L \) mean?).
(ii) Prove that
\[ \lim_{n \to \infty} \left( \frac{3}{4} \right)^n = 0 . \]
(iii) Prove that
\[
\lim_{n \to \infty} \frac{n^3 - 1}{n^3} = 1.
\]

Solution. (i) The statement “\(\lim_{n \to \infty} f(n) = L\)” means “For every \(\varepsilon > 0\), there exists \(N > 0\) such that if \(n \geq N\) then \(|f(n) - L| < \varepsilon\).”

(ii) Let \(\varepsilon > 0\). Set \(N = \lceil \log_{3/4} \varepsilon \rceil + 1\). Then for \(n \geq N\), we have \(n > \log_{3/4} \varepsilon\) so \((2/3)^n < \varepsilon\) (the direction of the inequality is switched because \(3/4 < 1\)). So \(|f(n) - L| = |(2/3)^n - 0| = (2/3)^n < \varepsilon\).

(iii) Let \(\varepsilon > 0\). Set \(N = \lceil \frac{1}{\sqrt{3}} \rceil + 1\). If \(n \geq N\), then \(n > \frac{1}{\sqrt{3}}\), so \(\frac{1}{n^3} < \varepsilon\). Thus \(|n^3 - 1 - 1| = \frac{1}{n^3} < \varepsilon\).

\(\square\)

Problem 3. (35 points)

(i) State the definition of limit for functions (i.e. what exactly does \(\lim_{x \to a} f(x) = L\) mean?)

(ii) Let \(f(x) = \sqrt{x - 3}\). Find a real number \(\delta\) such that the following is true: if \(x\) is a real number such that \(0 < |x - 7| < \delta\), then \(|f(x) - 2| < \frac{1}{3}\).

(iii) Prove that
\[
\lim_{x \to 0} x^{43} = 0.
\]

(iv) Prove that
\[
\lim_{x \to 3} x^2 - 4x = -3.
\]

Solution. (i) The statement “\(\lim_{x \to a} f(x) = L\)” means “For every \(\varepsilon > 0\), there exists \(\delta > 0\) such that if \(0 < |x - a| < \delta\) then \(|f(x) - L| < \varepsilon\).”

(ii) Here, \(f(x) = \sqrt{x - 3}, a = 7,\) and \(L = 2\). We have
\[
|f(x) - 2| < \frac{1}{3}
\]
\[
\iff 2 - \frac{1}{3} < \sqrt{x - 3} < 2 + \frac{1}{3}
\]
\[
\iff \frac{25}{9} < x - 3 < \frac{49}{9}
\]
\[
\iff -\frac{11}{9} < x - 7 < \frac{13}{9}
\]
so any value of \(\delta\) satisfying \(\delta < \min\{1, \frac{25}{9}, \frac{49}{9}\}\) works.

(iii) Let \(\varepsilon > 0\). Set \(\delta = \frac{\varepsilon}{3}\). If \(0 < |x - 0| < \delta = \frac{\varepsilon}{3}\), then \(|x^{43} - 0| = |x|^{43} < \varepsilon\).

(iv) Let \(\varepsilon > 0\). We have \((x^2 - 4x) = (x - 3)(x - 1).\) Set \(\delta = \varepsilon + 1\). Suppose \(0 < |x - 3| < \delta\). Then \(3 - \delta < x < 3 + \delta\) implies \(2 - \delta < x - 1 < 2 + \delta\) implies \(|x - 1| < 2 + \delta = \varepsilon + 1\). Thus \(|(x^2 - 4x) - (3)| < |x - 3||x - 1| < \delta(2 + \delta) = (\varepsilon + 1)(\varepsilon + 1) = \varepsilon\).

(How to find \(\delta\): if \(|x - 3| < \delta\), then \(3 - \delta < x < 3 + \delta\), so \(2 - \delta < x - 1 < 2 + \delta\), so \(|x - 1| < 2 + \delta\). If we can find a \(\delta\) satisfying \(\delta(2 + \delta) < \varepsilon\), then \(0 < |x - 3| < \delta\) implies \(|(x^2 - 4x) - (3)| < |x - 3||x - 1| < \delta(2 + \delta) < \varepsilon\), so we will be done. We can complete the square in \(\delta(2 + \delta) < \varepsilon\) to get \((\delta + 1)^2 < \varepsilon + 1\), which is equivalent to \(\delta + 1 < \sqrt{\varepsilon + 1}\) and \(\delta < \sqrt{\varepsilon + 1} - 1\).)

\(\square\)
Problem 4. (10 points) Evaluate the following limits and justify each step by indicating the appropriate Limit Laws.

(i) \[ \lim_{x \to -2} \left( \frac{t^2 - 2}{2t^2 - 3t + 2} \right)^3 \]

(ii) \[ \lim_{x \to 2} \sqrt[3]{\frac{2x^2 + 1}{3x - 2}} \]

Solution. (i) We have

\[ \lim_{x \to -2} \left( \frac{t^2 - 2}{2t^2 - 3t + 2} \right)^3 = \left( \lim_{x \to -2} \frac{t^2 - 2}{2t^2 - 3t + 2} \right)^3 \] (Power Law)

\[ = \left( \frac{(-2)^2 - 2}{2(-2)^2 - 3(-2) + 2} \right)^3 \] (DSP for rational functions)

\[ = \frac{1}{512}. \]

(ii) We have

\[ \lim_{x \to 2} \sqrt[3]{\frac{2x^2 + 1}{3x - 2}} = \sqrt[3]{\lim_{x \to 2} \frac{2x^2 + 1}{3x - 2}} \] (Root Law)

\[ = \sqrt[3]{\frac{2(2)^2 + 1}{3(2) - 2}} \] (DSP for rational functions)

\[ = \frac{3}{2}. \]

Problem 5. (10 points)

(i) What exactly does it mean for a function \( f(x) \) to be continuous at the point \( x = a \)?

(ii) State the Intermediate Value Theorem.

(iii) Use it to show that the polynomial \( p(x) = x^2 - \pi x + 2 \) has a root between 0 and 1.

Solution. (i) The condition “\( f(x) \) is continuous at \( x = a \)” means “\( f(a) \) is defined, \( \lim_{x \to a} f(x) \) exists, and \( \lim_{x \to a} f(x) = f(a) \)”.

(ii) Suppose that \( f \) is continuous on the closed interval \([a, b]\) and let \( N \) be any number between \( f(a) \) and \( f(b) \), where \( f(a) \neq f(b) \). Then there exists a number \( c \) in \((a, b)\) such that \( f(c) = N \). (The expression “between \( f(a) \) and \( f(b) \), where \( f(a) \neq f(b) \)” means “in the interval \((f(a), f(b))\) if \( f(a) < f(b) \); in the interval \((f(b), f(a))\) if \( f(b) < f(a) \).”)

(iii) The polynomial \( p(x) \) is continuous at every \( x = a \). We have \( p(0) = 2 \) and \( p(1) = 3 - \pi < 0 \); thus 0 is a number between \( p(0) \) and \( p(1) \). Hence, by the Intermediate Value Theorem, there exists \( c \in (0, 1) \) such that \( p(c) = 0 \).