THE SUBSTITUTION RULE

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**Theorem 1** (Substitution Rule for Definite Integrals, page 411). If $g'$ is continuous on $[a,b]$ and $f$ is continuous on the range of $g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx .$$

**Exercise 1** (Section 5.5, #27). Find an antiderivative of the function $f(x) = (x^2 + 1)(x^3 + 3x)^4$.

**Solution.** Set $u(x) = x^3 + 3x$. Then $u'(x) = 3(x^2 + 1)$. Thus

$$\int (x^2 + 1)(x^3 + 3x)^4 dx = \int \frac{1}{3}(x^3 + 3x)^4(3x^2 + 3) dx$$

$$= \int \frac{1}{3}u^4 du$$

$$= \frac{1}{15}u^5 + c$$

$$= \frac{1}{15}(x^3 + 3x)^5 + c .$$

**Exercise 2** (Section 5.5, #32). Find an antiderivative of the function $f(x) = \frac{\sin(\ln x)}{x}$.

**Solution.** Set $u(x) = \ln x$. Then $u'(x) = \frac{1}{x}$. Thus

$$\int \frac{\sin(\ln x)}{x} dx = \int \sin u du$$

$$= -\cos(u) + c$$

$$= -\cos(\ln x) + c .$$

**Exercise 3** (Section 5.5, #36). Find an antiderivative of the function $f(x) = \frac{2^x}{2^x + 3}$.

**Solution.** Set $u(x) = 2^x + 3$. Then $u'(x) = (\ln 2)2^x$. Thus

$$\int \frac{2^x}{2^x + 3} dx = \int \frac{1}{\ln 2} \frac{1}{2^x + 3}((\ln 2)2^x) dx$$

$$= \int \frac{1}{\ln 2} du$$

$$= \frac{1}{\ln 2} \ln u + c$$

$$= \frac{1}{\ln 2} \ln(2^x + 3) + c .$$
Exercise 4 (Section 5.5, #48). Find an antiderivative of the function \( f(x) = \frac{x}{1+x^4} \).

Solution. Set \( u(x) = x^2 \). Then \( u'(x) = 2x \). Thus
\[
\int \frac{x}{1+x^4} \, dx = \int \frac{1}{2} \frac{1}{1+(x^2)^2} (2x) \, dx \\
= \int \frac{1}{2} \frac{1}{1+u^2} \, du \\
= \frac{1}{2} \arctan(u) + c \\
= \frac{1}{2} \arctan(x^2) + c.
\]
\[
\square
\]

Exercise 5 (Section 5.5, #60). Evaluate the definite integral
\[
\int_0^1 xe^{-x^2} \, dx.
\]

Solution. Set \( f(x)xe^{-x^2} \). We have \( f(x) = -\frac{1}{2}e^uu' \) where \( u(x) = -x^2 \). Thus \(-\frac{1}{2}e^u = -\frac{1}{2}e^{-x^2}\) is an antiderivative of \( f(x) \). By the Fundamental Theorem of Calculus, we have
\[
\int_0^1 xe^{-x^2} \, dx = \left( -\frac{1}{2}e^{-1^2} \right) - \left( -\frac{1}{2}e^{-0^2} \right) = -\frac{1}{2}e^{-1} + \frac{1}{2}.
\]
\[
\square
\]

Exercise 6 (Section 5.5, #69). Evaluate the definite integral
\[
\int_e^{e^4} \frac{1}{x\sqrt{\ln x}} \, dx.
\]

Solution. Set \( f(x) = \frac{1}{x\sqrt{\ln x}} \). We have \( f(x) = \frac{1}{\sqrt{u}}u' \) where \( u(x) = \ln x \). Thus \( 2\sqrt{u} = 2\sqrt{\ln x} \) is an antiderivative of \( f(x) \). By the Fundamental Theorem of Calculus, we have
\[
\int_e^{e^4} \frac{1}{x\sqrt{\ln x}} \, dx = (2\sqrt{\ln e^4}) - (2\sqrt{\ln e}) = (2 - 2 - 2) = 2.
\]
\[
\square
\]

Exercise 7 (Section 5.5, #86). If \( f \) is continuous and \( \int_0^9 f(x) \, dx = 4 \), find \( \int_0^3 xf(x^2) \, dx \).

Solution. Set \( h(x) = xf(x^2) \). Then \( h(x) = \frac{1}{2}f(g(x))g'(x) \) where \( g(x) = x^2 \). Thus
\[
\int_0^3 h(x) \, dx = \int_0^3 \frac{1}{2}f(g(x))g'(x) \, dx = \int_{g(0)}^{g(3)} \frac{1}{2}f(x) \, dx = \frac{1}{2} \int_0^9 f(x) \, dx = \frac{1}{2} \cdot 4 = 2.
\]
\[
\square
\]

Exercise 8 (Section 5.5, #89). If \( a \) and \( b \) are positive numbers, show that
\[
\int_0^1 x^a(1-x)^b \, dx = \int_0^1 x^b(1-x)^a \, dx.
\]
Proof. Set  \( f(x) = x^a(1 - x)^b \). Let  \( g(x) = 1 - x \). Then

\[
\int_0^1 x^a(1 - x)^b \, dx = - \int_{g(0)}^{g(1)} x^a(1 - x)^b \, dx
\]

\[
= - \int_0^1 (1 - x)^a (1 - (1 - x))^b (-1) \, dx
\]

\[
= \int_0^1 (1 - x)^a x^b \, dx.
\]

\( \square \)

**Exercise 9** (Section 5.5, #90,91). If  \( f \) is continuous on  \([0, \pi]\), use the substitution  \( u = \pi - x \) to show that

\[
\int_0^\pi x f(\sin x) \, dx = \frac{\pi}{2} \int_0^\pi f(\sin x) \, dx.
\]

Use this to evaluate the integral

\[
\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} \, dx.
\]

**Solution.** We have

\[
\int_0^\pi x f(\sin x) \, dx = - \int_{u(0)}^{u(\pi)} x f(\sin x) \, dx
\]

\[
= - \int_0^\pi (\pi - x) f(\sin(\pi - x))(-1) \, dx
\]

\[
= \int_0^\pi (\pi - x) f(\sin x) \, dx \quad \text{since} \quad \sin(\pi - x) = \sin x
\]

\[
= \pi \int_0^\pi f(\sin x) \, dx - \int_0^\pi x f(\sin x) \, dx
\]

so

\[
\int_0^\pi x f(\sin x) \, dx = \frac{\pi}{2} \int_0^\pi f(\sin x) \, dx.
\]

Set  \( f(x) = \frac{\pi}{x - x^2} \). Then

\[
\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} \, dx = \int_0^\pi \frac{x \sin x}{2 - \sin^2 x} \, dx = \frac{\pi}{2} \int_0^\pi f(\sin x) \, dx = \frac{\pi}{2} \int_0^\pi \frac{\sin x}{1 + \cos^2 x} \, dx.
\]

We can compute the last integral by substitution. Set  \( u(x) = \cos x \). Then

\[
\int_0^\pi \frac{\sin x}{1 + \cos^2 x} \, dx = \int_0^\pi \frac{-1}{1 + (\cos x)^2} (- \sin x) \, dx
\]

\[
= \int_{u(0)}^{u(\pi)} - \arctan u \, du
\]

\[
= \int_{-1}^1 \arctan u \, du
\]

\[
= \frac{\pi}{2}.
\]

Thus

\[
\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} \, dx = \left( \frac{\pi}{2} \right)^2.
\]

\( \square \)