THE SUBSTITUTION RULE

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\textbf{Theorem 1} (Substitution Rule for Definite Integrals, page 411). If \( g' \) is continuous on \([a, b]\) and \( f \) is continuous on the range of \( g(x) \), then

\[ \int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx . \tag{1} \]

\textit{Proof.} Let \( F \) be an antiderivative of \( f \), i.e. we have \( F' = f \). By FTC(ii), we have

\[ F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(x)dx . \tag{2} \]

(You should read the LHS as the net change of \( F(x) \) between \( g(a) \) and \( g(b) \).) Also, notice that

\[ (F(g(x)))' = F'(g(x))g'(x) = f(g(x))g'(x) . \]

Thus \( F(g(x)) \) is an antiderivative of \( f(g(x))g'(x) \). By FTC(ii), we have

\[ F(g(b)) - F(g(a)) = \int_a^b f(g(x))g'(x)dx . \tag{3} \]

(You should read the LHS as the net change of \( F(g(x)) \) between \( a \) and \( b \).) Then (2) and (3) implies (1).\footnote{Equation (1) is false if we remove the limits of integration. The assertion \( \int f(g(x))g'(x)dx = \int f(x)dx \) seems to imply the following: “if \( F_1(x) \) is an antiderivative of \( f(g(x))g'(x) \) and \( F_2(x) \) is an antiderivative of \( f(x) \), then there exists a constant \( C \) such that \( F_1(x) = F_2(x) + C. \)” But this is false: consider \( f(x) = x^2 \) and \( g(x) = x^2 \). Then \( f(g(x))g'(x) = 2x^3 \). Let’s pick the antiderivatives \( F_1(x) = \frac{2}{3}x^3 \) and \( F_2(x) = \frac{1}{3}x^3 \) of \( f(g(x))g'(x) \) and \( f(x) \), respectively. Notice that there is no constant \( C \) such that \( F_1(x) = F_2(x) + C \) for all \( x \), since this says that the polynomial \( \frac{2}{3}x^3 - \frac{1}{3}x^3 - C \) has infinitely many roots. The correct statement is: “if \( F_1(x) \) is an antiderivative of \( f(g(x))g'(x) \) and \( F_2(x) \) is an antiderivative of \( f(x) \), then there exists a constant \( C \) such that \( F_1(x) = F_2(g(x)) + C. \)” (The only difference is that we have \( F_1(x) = F_2(g(x)) + C \) instead of \( F_1(x) = F_2(x) + C. \).) You can check this for the example \( f(x) = x^2 \) and \( g(x) = x^2 \).}

The Substitution Rule for Definite Integrals asserts that two numbers are equal. By fixing \( a \) and varying \( b \) in (1), we obtain an equality of functions

\[ G_1(t) = G_2(t) \]

where

\[ G_1(t) := \int_a^t f(g(x))g'(x)dx \quad \text{and} \quad G_2(t) := \int_{g(a)}^{g(t)} f(x)dx . \]

You can use the Substitution Rule backwards and forwards. It’s useful to know both methods.

\textbf{Substitution Method 1.} Given a function \( f(x) \) of which you want to find an antiderivative, find a function \( g(x) \) such that the function \( f(g(x))g'(x) \) has a known antiderivative, find an antiderivative \( G(x) \) of \( f(g(x))g'(x) \), and define \( F(x) := G(g^{-1}(x)) \). Then \( F(x) \) is an antiderivative of \( f(x) \). You can check as
follows:

\[ F'(x) = G'(g^{-1}(x))g'(-1)(x) \]
\[ = f(g(g^{-1}(x))) \cdot g'(g^{-1}(x)) \cdot \frac{1}{g'(g^{-1}(x))} \]
\[ = f(x) . \]

Let’s look at an example from class today.

**Problem 1.** Find an antiderivative of \( f(x) = e^{\sqrt{x}}. \)

**Solution.** Let \( g(x) = x^{2}. \) Then \( f(g(x))g'(x) = e^{\sqrt{x}} \cdot 2x = 2xe^{x}. \) An antiderivative of \( xe^{x} \) is \( xe^{x} - e^{x} \) (which you can find by integration by parts; set \( u(x) = x, \) \( v(x) = e^{x} \) so set \( G(x) = 2xe^{x} - 2e^{x}. \) Then \( G'(x) = f(g(x))g'(x). \) Set \( F(x) = G(g^{-1}(x)) = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}}. \) Then \( F(x) \) is an antiderivative of \( f(x). \) \( \square \)

**Substitution Method 2.** Given a function \( h(x) \) of which you want to find an antiderivative, find functions \( f(x) \) and \( g(x) \) such that \( h(x) = f(g(x))g'(x) \) and \( f(x) \) has a known antiderivative \( F(x). \) Then \( h(x) = f(g(x))g'(x) = F'(g(x))g'(x) = (F(g(x)))', \) so \( F(g(x)) \) is an antiderivative of \( h(x). \)

**Problem 2.** Find an antiderivative of \( h(x) = \tan x. \)

**Solution.** Write \( h(x) = \tan x = \frac{\sin x}{\cos x} \). Define \( f(x) = \frac{1}{x} \) and \( g(x) = \cos x. \) Then \( h(x) = \frac{1}{\cos x}(-\sin x) = f(g(x))g'(x). \) Since \(-\ln x \) is an antiderivative of \( f(x), -\ln(\cos x) \) is an antiderivative of \( h(x). \) \( \square \)

**Exercise 1** (Section 5.5, #27). Find an antiderivative of the function \( f(x) = (x^{2} + 1)(x^{3} + 3x)^{4}. \)

**Exercise 2** (Section 5.5, #32). Find an antiderivative of the function \( f(x) = \frac{\sin(x)}{x}. \)

**Exercise 3** (Section 5.5, #36). Find an antiderivative of the function \( f(x) = \frac{2x^{7}}{2x^{3} + 1}. \)

**Exercise 4** (Section 5.5, #48). Find an antiderivative of the function \( f(x) = \frac{x}{1+x^{2}}. \)

**Exercise 5** (Section 5.5, #60). Evaluate the definite integral

\[ \int_{0}^{1} xe^{-x^{2}} \, dx . \]

**Exercise 6** (Section 5.5, #69). Evaluate the definite integral

\[ \int_{e}^{e^{4}} \frac{1}{x \sqrt{\ln x}} \, dx . \]

**Exercise 7** (Section 5.5, #86). If \( f \) is continuous and \( \int_{0}^{a} f(x) \, dx = 4, \) find \( \int_{0}^{3} x f(x^{2}) \, dx. \)

**Exercise 8** (Section 5.5, #89). If \( a \) and \( b \) are positive numbers, show that

\[ \int_{0}^{1} x^{a}(1-x)^{b} \, dx = \int_{0}^{1} x^{b}(1-x)^{a} \, dx . \]

**Exercise 9** (Section 5.5, #90,91). If \( f \) is continuous on \([0, \pi] \), use the substitution \( u = \pi - x \) to show that

\[ \int_{0}^{\pi} x f(\sin x) \, dx = \frac{\pi}{2} \int_{0}^{\pi} f(\sin x) \, dx . \]

Use this to evaluate the integral

\[ \int_{0}^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx . \]