INDEFINITE INTEGRALS; FUNDAMENTAL THEOREM OF CALCULUS

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Summary of Fundamental Theorem of Calculus:

(1) Part 1 says that one (of many) antiderivatives of \( f(x) \) is given by the function \( F(x) = \int_a^x f(t) \, dt \). For this antiderivative, it is clear by the additive properties of the integral that 
\[ F(b) - F(a) = \int_a^b f(t) \, dt. \]

(2) Part 2 says that if \( G(x) \) is any other antiderivative of \( f(x) \), then 
\[ G(b) - G(a) = \int_a^b f(t) \, dt \] also.

Exercise 1 (Section 5.3, #9). Use Part 1 of FTC to find the derivative of the function 
\[ g(s) = \int_s^5 (t - t^2)^8 \, dt. \]

Solution. This is a straightforward application of FTC, which says that 
\( g(x) \) is an antiderivative of the function \((x - x^2)^8\), or in other words 
\( g'(x) = (x - x^2)^8 \).

Exercise 2 (Section 5.3, #13). Use Part 1 of FTC to find the derivative of the function 
\[ h(x) = \int_1^{e^x} \ln t \, dt. \]

Solution. Notice that \( h(x) \) is the composite of two functions \( f(x) = \int_x^1 \ln t \, dt \) and \( g(x) = e^x \), i.e. \( h(x) = f(g(x)) \). We know by FTC (part 1) that \( f'(x) = \ln x \). So, by the Chain Rule, we have 
\[ f'(x) = h'(g(x))g'(x) = \ln(e^x)e^x = xe^x. \]

(By the way, how can you use FTC (part 2) to solve this? Note that \( F(x) = x \ln x - x \) is an antiderivative of \( \ln x \). Thus \( h(x) = F(e^x) - F(1) = (e^x \ln(e^x)) - (1 \ln 1 - 1) = xe^x - e^x + 1 \). Thus \( h'(x) = xe^x \).)

Exercise 3 (Section 5.3, #31). Evaluate the integral 
\[ \int_0^{\pi/4} (\sec t)^2 \, dt. \]

Solution. Look up the fact that an antiderivative of \( (\sec x)^2 \) is \( \tan x \). By FTC (part 2), we have 
\[ \int_0^{\pi/4} (\sec t)^2 \, dt = \tan \frac{\pi}{4} - \tan 0 = 1 - 0 = 1. \]

Exercise 4 (Section 5.3, #40). Evaluate the integral 
\[ \int_1^2 \frac{4 + u^2}{u^3} \, du. \]

Solution. Notice that an antiderivative of \( \frac{4 + u^2}{u^3} \) is \( \frac{-2}{u^2} + \ln u \). By FTC (part 2), we have 
\[ \int_1^2 \frac{4 + u^2}{u^3} \, du = \left( \frac{-2}{u^2} - \ln 2 \right) - \left( \frac{-2}{1^2} - \ln 1 \right) = \frac{3}{2} - \ln 2. \]
Exercise 5 (Section 5.3, #72). If \( f \) is continuous and \( g \) and \( h \) are differentiable functions, find a formula for

\[
\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) \, dt .
\]

Solution. Let \( A(x) = \int_0^x f(t) \, dt \). By FTC (part 1), we have \( A'(x) = f(x) \). Then

\[
\int_{g(x)}^{h(x)} f(t) \, dt = \int_0^{h(x)} f(t) \, dt - \int_0^{g(x)} f(t) \, dt = A(h(x)) - A(g(x))
\]

so

\[
\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) \, dt = A'(h(x))h'(x) - A'(g(x))g'(x)
\]

\[
= f(h(x))h'(x) - f(g(x))g'(x) .
\]

\(\square\)