A definite integral is generally defined to be the limit of approximations of area. The formal definition is given on page 372, but it will usually suffice to use Theorem 4 on page 374, which says
\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^n f(x_i) \Delta x \quad \text{where} \quad \Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x .
\]
(1)

In other words, the integral \( \int_a^b f(x) \, dx \) is the limit of the sequence whose \( n \)th term is equal to the Riemann sum \( \sum_{i=1}^n f(x_i) \Delta x \), which in turn is the sum of the areas of \( n \) rectangles, where the \( i \)th rectangle has width \( \Delta x = \frac{b-a}{n} \) and (possibly negative) height \( f(x_i) = f(a + i\Delta x) \).

Exercise 1 (Section 5.2, #17). Express the limit \( \lim_{n \to \infty} \sum_{i=1}^n x_i \ln(1 + x_i^2) \Delta x \), where \( \Delta x = \frac{6-2}{n} \) and \( x_i = 2 + i\Delta x \), as a definite integral on the interval \([2, 6]\).

Solution. Set \( f(x) = x \ln(1 + x^2) \) and \( a = 2 \) and \( b = 6 \). Then
\[
\lim_{n \to \infty} \sum_{i=1}^n x_i \ln(1 + x_i^2) \Delta x = \int_2^6 x \ln(1 + x^2) \, dx .
\]

Exercise 2 (Section 5.2, #24). Use (1) to compute \( \int_0^2 (2x - x^3) \, dx \).

Solution. We have
\[
\int_0^2 (2x - x^3) \, dx = \lim_{n \to \infty} \sum_{i=1}^n \left( 2 \left( \frac{2i}{n} \right) - \left( \frac{2i}{n} \right)^3 \right) \frac{2}{n}
\]
\[
= \lim_{n \to \infty} \frac{8}{n^2} \sum_{i=1}^n i - \frac{16}{n^4} \sum_{i=1}^n i^3
\]
\[
= \lim_{n \to \infty} \frac{8}{n^2} \frac{n(n+1)}{2} - \frac{16}{n^4} \left( \frac{n(n+1)}{2} \right)^2
\]
\[
= \lim_{n \to \infty} \frac{4(n+1)}{n} - \frac{4(n+1)^2}{n^2}
\]
\[
= 0 .
\]

(You can check that you have the right answer using FTC. Since \( F(x) = x^2 - \frac{1}{4}x^4 \) is an antiderivative of \( 2x - x^3 \), FTC says \( \int_0^2 (2x - x^3) \, dx = F(2) - F(0) = 0 \)).

Exercise 3. Use (1) to compute \( \int_0^4 e^x \, dx \).
Solution. This turned out to be much harder than I expected. Don’t expect anything like this to show up on the midterms or final. For those interested, I’ll provide a proof. We have

$$\int_0^4 e^x \, dx = \lim_{n \to \infty} \sum_{i=1}^n e^{i \cdot \frac{4}{n}} \cdot \frac{4}{n}$$

$$= \lim_{n \to \infty} \frac{4 \cdot e^{\frac{4}{n}} - e^{(n+1) \cdot \frac{4}{n}}}{n \cdot \frac{1 - e^{\frac{4}{n}}}{\frac{1 - e^{\frac{4}{n}}}{n}}}$$

$$= \lim_{n \to \infty} \frac{4 \cdot e^{\frac{4}{n}} (1 - e^{\frac{4}{n}})}{n \cdot \frac{1 - e^{\frac{4}{n}}}{n}}$$

$$= (1 - e^{\frac{4}{n}}) \lim_{n \to \infty} \frac{4 e^{\frac{4}{n}}}{1 - e^{\frac{4}{n}}}.$$ 

Let $f$ be the function

$$f(x) = \frac{4 e^{\frac{4}{x}}}{1 - e^{\frac{4}{x}}}$$

with domain of definition $(0, \infty)$. I want to find the limit (of sequences) $\lim_{n \to \infty} f(n)$. I claim that this is equal to the limit (of functions) $\lim_{x \to \infty} f(x)$. This depends crucially on the fact that the function $f(x)$ is monotonically increasing on $(0, \infty)$, which you can check by showing that the derivative is always greater than 0:

$$f'(x) = \left(\frac{-4}{x^2} e^{\frac{4}{x}} + \frac{4}{x} e^{\frac{4}{x}} \cdot (-\frac{4}{x^2})\right) \left(1 - e^{\frac{4}{x}}\right) - \left(\frac{4}{x} e^{\frac{4}{x}}\right) \left(-e^{\frac{4}{x}} \cdot (-\frac{4}{x^2})\right)$$

$$= \frac{-4}{x^2} e^{\frac{4}{x}} \left(1 + \frac{4}{x}\right) \left(1 - e^{\frac{4}{x}}\right) + \frac{4}{x} e^{\frac{4}{x}}$$

$$= \frac{4}{x^2} e^{\frac{4}{x}} \left(1 - e^{\frac{4}{x}}\right)^2 \left(e^{\frac{4}{x}} - 1 - \frac{4}{x}\right)$$

where in the last term, the first factor is always positive. The second factor is always positive since, for any $\alpha > 0$, we have $e^{\alpha} \geq 1 + \alpha$ (this is because $e^{\alpha} = 1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \cdots$). This shows that $f$ is monotonically increasing on $(0, \infty)$.

So suppose we know $\lim_{n \to \infty} f(n) = L$ and $L$ is finite. Let $\epsilon > 0$; then there exists $M > 0$ such that $n \geq M$ implies $|f(n) - L| < \epsilon$. Suppose $x > M$. Then there exists an integer $N$ such that $N \leq x \leq N + 1$. Since $|f(N) - L| < \epsilon$ and $|f(N + 1) - L| < \epsilon$ and $f(N) \leq f(x) \leq f(N + 1)$, we have $|f(x) - L| < \epsilon$. Thus $\lim_{x \to \infty} f(x) = L$.

Conversely, suppose we know $\lim_{x \to \infty} f(x) = L$ and $L$ is finite. Let $\epsilon > 0$; then there exists $M > 0$ such that $x > M$ implies $|f(x) - L| < \epsilon$. Let $N = \lfloor M \rfloor$. For any $n > N$, we have $n > M$, so $|f(n) - L| < \epsilon$. Thus $\lim_{n \to \infty} f(n) = L$.

The proofs of the statements “$\lim_{n \to \infty} f(n) = \infty$ if and only if $\lim_{x \to \infty} f(x) = \infty$” and “$\lim_{n \to \infty} f(n) = -\infty$ if and only if $\lim_{x \to \infty} f(x) = -\infty$” are similar.

So let’s compute $\lim_{x \to \infty} f(x)$. Since we actually have a function instead of a sequence, we can use L’Hospital’s Rule (it turns out that L’Hospital’s Rule does include the case when $x$ approaches $\infty$; you can
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look at Note 2 at the top of page 303). We have

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{-\frac{4}{x^2} e^{\frac{4}{x}} + \frac{4}{x} e^{\frac{4}{x}} (-\frac{4}{x^2})}{- \frac{1}{x^2} (-\frac{4}{x^2})}
\]

\[
= \lim_{x \to \infty} \frac{1 + \frac{4}{x}}{-1}
\]

= -1.

Hence we have

\[
\int_0^4 e^x \, dx = (1 - e^4) \lim_{n \to \infty} \frac{\frac{4}{n} e^{\frac{4}{n}}}{1 - e^{\frac{4}{n}}}
\]

\[
= (1 - e^4) \lim_{x \to \infty} f(x)
\]

\[
= (1 - e^4)(-1)
\]

\[
= e^4 - 1.
\]

\[\square\]

Exercise 4 (Section 5.2, #27). Prove that \( \int_a^b x \, dx = \frac{1}{2} (b^2 - a^2) \). Note that this is \( f(b) - f(a) \) where \( f(x) = \frac{1}{2} x^2 \).

Solution. We have

\[
\int_a^b x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left( a + \frac{b - a}{n} \right) \frac{b - a}{n}
\]

\[
= \lim_{n \to \infty} \frac{b - a}{n} \sum_{i=1}^{n} 1 + \left( \frac{b - a}{n} \right)^2 \sum_{i=1}^{n} i
\]

\[
= \lim_{n \to \infty} \frac{b - a}{n} (n) + \left( \frac{b - a}{n} \right)^2 \frac{n(n+1)}{2}
\]

\[
= a(b - a) + (b - a)^2 \frac{1}{2}
\]

\[
= \frac{1}{2} (b^2 - a^2)
\]

\[\square\]

Exercise 5 (Section 5.2, #28). Prove that \( \int_a^b x^2 \, dx = \frac{1}{3} (b^3 - a^3) \). Note that this is \( f(b) - f(a) \) where \( f(x) = \frac{1}{3} x^3 \).
Solution. We have
\[
\int_a^b x^2 \, dx = \lim_{n \to \infty} \sum_{i=1}^n \left( a + \frac{i(b-a)}{n} \right)^2 \frac{b-a}{n}
\]
\[
= \lim_{n \to \infty} \frac{b-a}{n^3} \sum_{i=1}^n (an + i(b-a))^2
\]
\[
= \lim_{n \to \infty} \frac{b-a}{n^3} \sum_{i=1}^n \left( a^2n^2 + 2ain(b-a) + i^2(b-a)^2 \right)
\]
\[
= \lim_{n \to \infty} \frac{b-a}{n^3} \left( a^2 n^2 \sum_{i=1}^n 1 + 2an(b-a) \sum_{i=1}^n i + (b-a)^2 \sum_{i=1}^n i^2 \right)
\]
\[
= \lim_{n \to \infty} \frac{b-a}{n^3} \left( a^2 n^2 \frac{n(n+1)}{2} + (b-a)^2 \frac{n(n+1)(2n+1)}{6} \right)
\]
\[
= (b-a) \left( a^2 + 2a(b-a) \frac{1}{2} + \frac{(b-a)^2}{3} \right)
\]
\[
= \frac{1}{3} (b^3 - a^3).
\]

Exercise 6 (Section 5.2, #29). Express the integral \( \int_1^{10} (x - 4 \ln x) \, dx \) as a limit of Riemann sums as in  \( \text{[1]} \).

Solution. We have
\[
\int_1^{10} (x - 4 \ln x) \, dx = \lim_{n \to \infty} \sum_{i=1}^n \left( 1 + i \frac{10 - 1}{n} - 4 \ln \left( 1 + i \frac{10 - 1}{n} \right) \right) \frac{10 - 1}{n}.
\]

Exercise 7 (Section 5.2, #57). Use the properties of integrals (page 379 to 381) to verify the following inequality without evaluating an integral:
\[
2 \leq \int_{-1}^{1} \sqrt{1 + x^2} \, dx \leq 2\sqrt{2}.
\]

Solution. Since \( 0 \leq x^2 \leq 1 \) for all \( x \in [-1, 1] \), we have \( 1 \leq \sqrt{1 + x^2} \leq \sqrt{2} \) for all \( x \in [-1, 1] \). Thus
\[
2 = \int_{-1}^{1} 1 \, dx \leq \int_{-1}^{1} \sqrt{1 + x^2} \, dx \leq \int_{-1}^{1} \sqrt{2} \, dx = 2\sqrt{2}
\]
where in the inequalities marked by \( \text{(*)} \) we used property 7 on page 381. Alternatively, you can obtain the result straight from property 8 on the same page.