MORE FUNCTION GRAPHING; OPTIMIZATION

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**Exercise 1.** Let $n$ be an arbitrary positive integer. Give an example of a function with exactly $n$ vertical asymptotes. Give an example of a function with infinitely many vertical asymptotes.

**Solution.** The function $y = \frac{1}{(x-1)(x-2)\cdots(x-n)}$, with domain of definition $\mathbb{R} \setminus \{1,2,\ldots,n\}$, has exactly $n$ vertical asymptotes, namely at $x = 1, \ldots, n$. In general, given $n$ distinct numbers $a_1, \ldots, a_n$, the function $y = \frac{1}{(x-a_1)(x-a_2)\cdots(x-a_n)}$, with domain of definition $\mathbb{R} \setminus \{a_1,\ldots,a_n\}$, has exactly $n$ vertical asymptotes, namely at $x = a_1, \ldots, a_n$.

The function $y = \tan x$, with domain of definition $\mathbb{R} \setminus \{\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\}$, has infinitely many vertical asymptotes, exactly at $x \in \{\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\}$, which is where $\cos x = 0$. $\square$

**Exercise 2.** Let $f$ be a function which is differentiable everywhere. Suppose that $f'(x) > 1$ for all $x$. Show that $\lim_{x \to \infty} f(x) = \infty$.

**Solution.** Let $x_1 < x_2$ be real numbers. By the Mean Value Theorem, there exists some $c \in (x_1, x_2)$ such that $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$, and since $f'(c) > 1$ we have $f(x_2) - f(x_1) > x_2 - x_1$. In particular, substituting $x_2 = x$ and $x_1 = 0$, we have $f(x) > x + f(0)$ for all $x > 0$. Thus, since $\lim_{x \to \infty}(x + f(0)) = \infty$, we have $\lim_{x \to \infty} f(x) = \infty$.

An alternate solution is to use the Fundamental Theorem of Calculus. We have

$$f(x) - f(0) = \int_0^x f'(t) \, dt > \int_0^x 1 \, dt = x$$

where we used FTC in the equality marked $(\ast)$ above.

Be careful about using indefinite integrals. It is not necessarily true that $f(x) > x$ for all $x$; it is only true for all $x$ greater than some number. For example, consider the case $f(x) = 2x$; then $f'(x) = 2$ so $f'(x) > 1$ for all $x$. But $f(x) > x$ only if $x > 0$. $\square$

**Exercise 3.** Graph the function

$$f(x) = x^3 + 6x^2 + 9x.$$

Indicate domain, critical points, inflection points, regions where the graph is increasing/decreasing, $x$-intercepts and $y$-intercepts, regions of concavity (up or down), local maxima and minima, any asymptotes and behavior at infinity.

**Solution.** The domain is $\mathbb{R}$. A point $(x, f(x))$ on the graph of the function is a critical point if $f'(x) = 0$ or is not defined; $f(x)$ is differentiable everywhere, so the only way a point can be a critical point is if $f'(x) = 0$. We have

$$f'(x) = 3x^2 + 12x + 9 = 3(x+1)(x+3)$$

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so the only critical points of \( f \) are \((-1, f(-1)) = (-1, -4) \) and \((-3, f(-3)) = (-3, 0) \). A point \((x, f(x))\) on the graph of the function is an inflection point if \( f''(x) = 0 \) or is not defined; again, the only way a point can be an inflection point is if \( f''(x) = 0 \). We have

\[
f''(x) = 6x + 12
\]

so the only critical point of \( f \) is \((-2, f(-2)) = (-2) \). We can factor \( f(x) = (x+3)^2 \), so the \( x \)-intercepts are \( x = 0, -3 \). The \( y \)-intercept is \( f(0) = 0 \).

The function \( f \) is increasing (resp. decreasing) in the interval \( I \) if \( f(x_1) < f(x_2) \) (resp. \( f(x_1) > f(x_2) \)) for all \( x_1, x_2 \in I \) such that \( x_1 < x_2 \) (this is the definition on page 19). This is equivalent to saying that \( f'(x) > 0 \) (or \( f'(x) < 0 \)) for all \( x \) in the interval\(^1\) Since \( f'(x) > 0 \) on \((\infty, -3) \) and \((-1, \infty) \) and \( f'(x) < 0 \) on \((-3, -1) \), \( f \) is increasing on \((\infty, -3) \), decreasing on \((-3, -1) \), and increasing on \((-1, \infty) \).

A function \( f \) is concave up (resp. down) on the interval \( I \) if \( f''(x) > 0 \) (resp. \( f''(x) < 0 \)) for all \( x \in I \).

Since \( f''(x) = 6(x+2) \), \( f''(x) > 0 \) on the interval \((-2, \infty) \) and \( f''(x) < 0 \) on the interval \((-\infty, -2) \). So \( f \) is concave up on \((-2, \infty) \) and concave down on \((-\infty, -2) \). A function which is differentiable on all of \( \mathbb{R} \) has a local maximum (resp. minimum) at \( x = a \) if and only if \( f'(a) = 0 \) and \( f''(a) < 0 \) (resp. \( f''(a) > 0 \)). We have \( f'(-1) = 0 \) and \( f''(-1) = 6 > 0 \), so \( f \) has a local minimum at \( x = -1 \); we have \( f'(-3) = 0 \) and \( f''(-3) = -6 < 0 \), so \( f \) has a local maximum at \( x = -3 \).

There are no vertical asymptotes because the function is continuous on the entire real line. Suppose there is a slant or horizontal asymptote; then there exist real numbers \( m, b \) such that either

\[
\lim_{x \to \infty} (f(x) - (mx + b)) = 0 \quad \text{or} \quad \lim_{x \to -\infty} (f(x) - (mx + b)) = 0.
\]

But \( f(x) - (mx + b) \) is a polynomial of degree 3, so we have a contradiction (recall the exercise about endpoint behavior of nonconstant polynomials). So there are no slant or horizontal asymptotes.

![Graph of \( f(x) = x^3 + 6x^2 + 9x \)](image_url)

\[\text{Figure 1. Graph of } f(x) = x^3 + 6x^2 + 9x \]

\(^1\)Notice that the function \( f(x) = \frac{1}{x} \) defined on \( \mathbb{R} \setminus \{0\} \) has the property that \( f'(x) < 0 \) for all \( x \in \mathbb{R} \setminus \{0\} \), but it is not decreasing on \( \mathbb{R} \setminus \{0\} \) (since \( f(-1) < f(1) \)). This has to do with the fact that \( f \) is not defined at \( x = 0 \).
Exercise 4. Find 
\[
\lim_{t \to 16} \frac{\sqrt{t} - 4}{t - 16}
\]
in three ways: (i) using methods learned up to and including the first midterm; (ii) by realizing the limit as \(f'(c)\) for some function \(f(t)\) and some value \(c\); (iii) using L’Hospital’s Rule.

Solution. (i) We have 
\[
\lim_{t \to 16} \frac{\sqrt{t} - 4}{t - 16} = \lim_{t \to 16} \frac{\sqrt{t} - 4}{(\sqrt{t} - 4)(\sqrt{t} + 4)} = \lim_{t \to 16} \frac{1}{\sqrt{t} + 4} = \frac{1}{8}.
\]

(ii) Set \(f(t) = \sqrt{t}\). Then 
\[
\lim_{t \to 16} \frac{\sqrt{t} - 4}{t - 16} = \lim_{t \to 16} \frac{f(t) - f(16)}{t - 16} = f'(16) = \frac{1/2}{\sqrt{16}} = \frac{1}{8}.
\]

(iii) We have 
\[
\lim_{t \to 16} \frac{\sqrt{t} - 4}{t - 16} = \lim_{t \to 16} \frac{\log t}{t} = \frac{1}{8}.
\]

\[\blacksquare\]

Exercise 5 (Section 4.7, #19). Find the point on the line \(y = 2x + 3\) that is closest to the origin.

Solution. At the point \((x, 2x + 3)\), the distance to the origin is 
\[
d(x) = \sqrt{(x - 0)^2 + (2x + 3 - 0)^2} = \sqrt{x^2 + (2x + 3)^2} = \sqrt{5x^2 + 12x + 9}
\]

by the Pythagorean theorem. The function \(d(x)\) is defined for all \(x\) and is differentiable everywhere. We want to find a global minimum of \(d(x)\). To ease computation, I make the following (perhaps unconventional) argument. Let \(p(x) = 5x^2 + 12x + 9\). For any two nonnegative real numbers, the condition \(\sqrt{x_1} \leq \sqrt{x_2}\) is equivalent to \(x_1 \leq x_2\). Thus, for any two real numbers \(x_1, x_2\), the condition \(d(x_1) \leq d(x_2)\) is equivalent to \(p(x_1) \leq p(x_2)\) (since \(d(x) = \sqrt{p(x)}\)). Thus \(d(x)\) is a global minimum value of the function \(d(x)\) if and only if \(p(x)\) is a global minimum value of the function \(p(x)\). Let’s find the global minimum value of \(p(x)\). For quadratic polynomials \(ax^2 + bx + c\) with \(a > 0\), the minimum value takes place at \(x = -\frac{b}{2a}\) since that’s the vertex of the parabola. So in our case the minimum of \(p(x)\) occurs at \(x = -\frac{12}{2 \cdot 5} = -\frac{6}{5}\). Thus the desired point is \((-\frac{6}{5}, 2(-\frac{6}{5}) + 3) = (-\frac{6}{5}, -\frac{3}{5})\).

\[\blacksquare\]

Exercise 6 (Section 4.7, #21). Find the points on the ellipse \(4x^2 + y^2 = 4\) that are farthest away from the point \((1, 0)\).

Solution. I’m going to find the point on the upper half of the ellipse which is farthest from the point \((1, 0)\), then note that the ellipse is symmetric with respect to the \(x\)-axis, so the reflection will be farthest, too. The equation of the upper part is given by \(y = \sqrt{4 - 4x^2}\), with domain of definition \([-1, 1]\). It is continuous on \([-1, 1]\) and differentiable on \((-1, 1)\). The distance from the point \((x, \sqrt{4 - 4x^2})\) to \((1, 0)\) is given by 
\[
d(x) = \sqrt{(x - 1)^2 + (\sqrt{4 - 4x^2} - 0)^2} = \sqrt{-3x^2 - 2x + 5}
\]

by the Pythagorean theorem. It is continuous on \([-1, 1]\) and differentiable on \((-1, 1)\). Set \(p(x) = -3x^2 - 2x + 5\). By the argument given in the solution to Exercise 5, the problem reduces to finding the maximum

\[\text{Exercise: Prove this.}\]

\[\text{Note that the line joining } (-\frac{6}{5}, -\frac{3}{5}) \text{ to the origin is perpendicular to the line } y = 2x + 3. \text{ This will always be the case for problems of the form “find the point on the line } \ell \text{ that is closest to the point } P\text{.”}\]
of the quadratic $p(x)$ on the interval $[-1,1]$. If the quadratic were defined over all of $\mathbb{R}$, then its global maximum would occur at $x = -\frac{a^2}{2} = -\frac{3}{2}$, which is inside the interval $[-1,1]$, so this a global maximum of the quadratic on the interval $[-1,1]$. Thus the point $(-\frac{1}{4}, \sqrt{4 - 4(-\frac{1}{4})^2}) = (-\frac{1}{4}, \frac{\sqrt{2}}{2})$ is the point on the upper half of the ellipse which is farthest away from $(1,0)$. On the bottom half, the point $(\frac{1}{4}, -\frac{\sqrt{2}}{2})$ is farthest away. So the desired points are $(-\frac{1}{4}, \frac{\sqrt{2}}{2})$ and $(-\frac{1}{4}, -\frac{\sqrt{2}}{2})$. □

**Exercise 7** (Section 4.7, #24). Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.

**Solution.** Assume without loss of generality that $a,b$ are positive. If $b = a$, then the ellipse is actually a circle and any rectangle inscribed in a circle of radius $a$ is a square of side length $a\sqrt{2}$, so it has area $2a^2$.

Now assume $b < a$. I’m going to assume without proof that, for any rectangle inscribed in an ellipse which is not a circle, its sides are parallel to the axes of the ellipse. Since the sides of our rectangles are parallel to the $x$ and $y$ axes, its vertices are $(x,y), (-x,y), (-x,-y), (x,-y)$ for some $x \in [0,a]$ and $y = b\sqrt{1 - \frac{x^2}{a^2}}$.

Thus the area of the rectangle is

$$A(x) = 2x \cdot 2y = 4bx \sqrt{1 - \frac{x^2}{a^2}}$$

where $A(x)$ is a function on the domain $[0,a]$. It is continuous on $[0,a]$ and differentiable on $(0,a)$. We maximize $A(x)$. Let $p(x) = x^2 - \frac{1}{2}x^4$. Then $A(x) = 4b\sqrt{p(x)}$. By the argument given in the solution to Exercise 5, the problem reduces to finding the maximum of $p(x)$ on the interval $[0,a]$. Considering $p(x)$ to be a quadratic polynomial in the variable $x^2$, we have that it takes a maximum when $x^2 = -\frac{1}{2 - \frac{a^2}{2}} = \frac{a^2}{2}$, or when $x = \frac{a}{\sqrt{2}}$, which is inside the interval $[0,a]$. For this value of $x$, we have $y = b\sqrt{1 - \frac{(\frac{a}{\sqrt{2}})^2}{a^2}} = \frac{b}{\sqrt{2}}$, so the rectangle has area $2 \frac{a}{\sqrt{2}} \cdot 2 \frac{b}{\sqrt{2}} = 2ab$.

**Exercise 8** (Section 4.7, #54). At which points on the curve $y = 1 + 40x^3 - 3x^5$ does the tangent line have the largest slope?

**Solution.** Let $f(x) = 1 + 40x^3 - 3x^5$. At $(x,f(x))$, the tangent line to the curve $y = f(x)$ has slope $f'(x) = 120x^2 - 15x^4$. So our problem reduces to maximizing $f'(x)$. We have $f''(x) = 240x - 60x^3 = 60x(x-2)(x+2)$, which is 0 only if $x = 0, \pm 2$. We have $f'''(x) = 240 - 180x^2$ and $f'''(0) = 240 > 0, f'''(-2) = -480 < 0, f'''(-2) = -480 < 0$, so 0 is a local minimum while $\pm 2$ are local maxima. Since $f''(x) > 0$ on the intervals $(-\infty, -2)$ and $(0, 2)$ and $f''(x) < 0$ on the intervals $(-2, 0)$ and $(2, \infty)$, we have that $f'(x)$ is increasing on the intervals $(-\infty, -2)$ and $(0, 2)$ and $f'(x)$ is decreasing on the intervals $(-2, 0)$ and $(2, \infty)$. Thus $(-2, f(-2)) = (-2, -223)$ and $(2, f(2)) = (2, 225)$ are the global maxima of $f'(x)$.

Alternatively, you can complete the square in $f'(x)$. We have $f'(x) = -15(x^4 - 8x^2)$. Completing the square gives $f'(x) = -15(x^4 - 8x^2 + 16) + 240 = -15(x^2 - 4)^2 + 240$, so $f'(x)$ takes its maximum whenever

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4This is a minor technical point which is less relevant to Math 1A but is still necessary in a complete solution: if a rectangle is inscribed in an ellipse which is not a circle, how do I know that its sides are going to be parallel to the $x$ and $y$ axes? (For example, if the ellipse is a circle, then inscribed rectangles are squares, whose sides don’t have to be parallel to the axes.) I hope this following argument will satisfy you. Assume without loss of generality that $b < a$ and a rectangle is inscribed in the ellipse such that its sides aren’t parallel to the axes of the ellipse. Apply the linear transformation which shrinks everything along the $x$-axis by the factor $k = \frac{b}{a}$. So the ellipse now becomes a circle of radius $b$. Under this transformation, the rectangle that was inscribed in the ellipse becomes a parallelogram. The four sides of the original rectangle have slope $m, -\frac{1}{m}, m, -\frac{1}{m}$ for some nonzero $m$, so the parallelogram has four sides whose slopes are $mk, -\frac{1}{mk}, mk, -\frac{1}{mk}$. Since $k > 1$, adjacent sides of the parallelogram aren’t perpendicular, and the parallelogram is not a rectangle. Thus the parallelogram has an acute angle, and such parallelograms cannot be inscribed in circles, contradiction.

5Note that this gives the same answer as above for the special case $b = a$. 
$(x^2 - 4)^2$ is minimized, i.e. when $x^2 - 4 = 0$, or $x = \pm 2$. Thus our points are $(-2, f(-2)) = (-2, -223)$ and $(2, f(2)) = (2, 225)$. □