Problem 1. (i) Complete the sentence: “A function \( f \) is continuous at \( x = a \) if . . .” Give an example of a function \( f \) and some \( a \) in the domain of \( f \) such that \( f \) is not continuous at \( x = a \).

(ii) Complete the sentence: “A function \( f \) is differentiable at \( x = a \) if . . .” Give an example of a function \( f \) and some \( a \) in the domain of \( f \) such that \( f \) is not differentiable at \( x = a \).

Solution. (i) A function \( f \) is continuous at \( x = a \) if \( \lim_{x \to a} f(x) = f(a) \). (For the question of continuity to make sense, the domain of definition of \( f \) needs to contain an open interval around \( a \).) The function

\[
f(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
-1 & \text{if } x < 0 
\end{cases}
\]

is not continuous at \( x = 0 \), since \( \lim_{x \to 0^+} f(x) = 1 \neq f(0) \) and \( \lim_{x \to 0^-} f(x) = 1 \neq f(0) \).

(ii) A function \( f \) is differentiable at \( x = a \) if the limit \( \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \) exists and is finite. (For the question of differentiability to make sense, the domain of definition of \( f \) needs to contain an open interval around \( a \).) The function \( f(x) = |x| \) is not differentiable at \( x = 0 \), since \( \frac{f(x+h)-f(x)}{h} \) is 1 if \( h > 0 \) and \( -1 \) if \( h < 0 \), so the two one-sided limits \( \lim_{h \to 0^+} \frac{f(x+h)-f(x)}{h} \) and \( \lim_{h \to 0^-} \frac{f(x+h)-f(x)}{h} \) do not agree.

Problem 2. (i) Find the equation of the tangent line to the curve \( y = \frac{2 \ln x}{x^2} - (\sin^{-1}(x))^5 \) at the point \( x = \frac{1}{2} \).

(ii) Find the tangent line to the graph \( x^2 + 2y^4 = 3 \) at the point \((1, 1)\).

Solution. (i) We have

\[
y \left( \frac{1}{2} \right) = \frac{2 \ln \frac{1}{2}}{(\frac{1}{2})^3} - \left( \sin^{-1} \frac{1}{2} \right)^5 = -16 \ln 2 - \left( \frac{\pi}{6} \right)^5
\]

and

\[
y'(x) = \frac{2}{x^4} - \frac{6 \ln x}{x^4} - 5 \left( \sin^{-1}(x) \right)^4 \frac{1}{\sqrt{1-x^2}}
\]

so

\[
y' \left( \frac{1}{2} \right) = 32 + 96 \ln 2 - 5 \left( \frac{\pi}{6} \right)^4 \frac{2}{\sqrt{3}}.
\]

Thus the equation of the tangent line at \( x = \frac{1}{2} \) is

\[
y - \left( -16 \ln 2 - \left( \frac{\pi}{6} \right)^5 \right) = \left( 32 + 96 \ln 2 - 5 \left( \frac{\pi}{6} \right)^4 \frac{2}{\sqrt{3}} \right) \left( x - \frac{1}{2} \right).
\]
(ii) Use implicit differentiation:

\[ 2x + 8y^3y' = 0 \implies y'(x) = -\frac{2x}{8y^3} \]

so \( y'(1) = -\frac{2(1)}{8(1)^3} = -\frac{1}{4} \). Thus the equation of the tangent line is

\[ y - 1 = -\frac{1}{4}(x - 1) \]

\[ \square \]

**Problem 3.**

(i) Let \( f(x) \) be a function of \( x \) such that \( (f(x))^x = 1 \). Find \( f'(x) \) in terms of \( f \).

(ii) Let \( f, g \) be differentiable functions such that \( g(x) > 0 \) for all \( x \). Set \( h(x) = f(x^g(x)) \). Find \( h' \) in terms of \( f, g, f', g' \).

(iii) Let \( f(x) = 3x \). Find a function \( F \) whose derivative is \( f \).

**Solution.**

(i) Taking natural logarithm of both sides, we have

\[ x \ln(f(x)) = 0 \]

Taking the derivative with respect to \( x \) gives

\[ \ln(f(x)) + x \frac{1}{f(x)} f'(x) = 0 \]

Thus

\[ f'(x) = -\frac{f(x) \ln(f(x))}{x} \]

(ii) Define \( k(x) = x^g(x) \). Then \( h(x) = f(k(x)) \). We have

\[ \ln(k(x)) = g(x) \ln x \]

so

\[ \frac{1}{k(x)} k'(x) = g'(x) \ln x + g(x) \frac{1}{x} \]

and

\[ k'(x) = x^g(x) \left( g'(x) \ln x + g(x) \frac{1}{x} \right) \]

thus

\[ h'(x) = f'(k(x)) \cdot k'(x) \]

\[ = f'(x^g(x)) \cdot x^g(x) \left( g'(x) \ln x + g(x) \frac{1}{x} \right) \]

(iii) An example is \( F(x) = \frac{3}{2}x^2 \).

\[ \square \]

**Problem 4.** Strontium-90 has a half-life of 28 days. A sample has a mass of 50 mg initially.

(a) How much of the sample remains after \( t \) days?

(b) How long does it take the sample to decay to a mass of 2 mg?
Solution. (a) Let \( y(t) \) be the quantity which remains after \( t \) days. Then there exists a constant \( C \) such that \( y(t) = y(0)e^{Ct} \). We know \( y(0) = 50 \) so \( y(t) = 50e^{Ct} \). Since the half-life is 28 days, we have \( y(28) = \frac{1}{2}y(0) = 25 \). Thus \( y(28) = y(0)e^{28C} \) implies \( e^{28C} = \frac{1}{2} \), or \( C = \frac{1}{28} \ln \frac{1}{2} \). We have \( e^{Ct} = (e^{\frac{1}{2} \ln \frac{1}{2}})^t = (\frac{1}{2})^t \) so \( y(t) = 50(\frac{1}{2})^t \).

(b) Let \( t_0 \) be the amount of time it takes for the sample to decay to 2 mg. We have \( 2 = y(t_0) = 50(\frac{1}{2})^{t_0} \) so \( (\frac{1}{2})^{t_0} = \frac{1}{25} \), which means \( t_0 = \frac{\ln 25}{\ln 2} \), and \( t_0 = \frac{28\ln 25}{\ln 2} \).

\[ \square \]

Problem 5. (i) State Rolle’s Theorem.

(ii) Use Rolle’s Theorem to prove the Mean Value Theorem.

(iii) Let \( f(x) = \frac{1}{5}x^5 - \frac{2}{5}x^3 + x \). Prove that \( f(\pi) > f(1) \).

Solution. (i) The statement of Rolle’s Theorem is:

**Theorem 1.** Let \( f \) be a function which is continuous on \([a, b] \) and differentiable on \((a, b) \). If \( f(a) = f(b) \), then there exists some \( c \in (a, b) \) such that \( f'(c) = 0 \).

(ii) We use Rolle’s Theorem to prove the Mean Value Theorem:

**Theorem 2.** Let \( f \) be a function which is continuous on \([a, b] \) and differentiable on \((a, b) \). Then there exists some \( c \in (a, b) \) such that \( f'(c) = \frac{f(b) - f(a)}{b-a} \).

**Proof.** Define the function \( g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b-a}(x-a) \). Then \( g \) is a function which is continuous on \([a, b] \) and differentiable on \((a, b) \). We also have \( g(a) = g(b) \). By Rolle’s Theorem, there exists some \( c \in (a, b) \) such that \( g'(c) = 0 \). We have \( g'(x) = f'(x) - \frac{f(b) - f(a)}{b-a} \). Thus \( f'(c) = \frac{f(b) - f(a)}{b-a} \).

\[ \square \]

(iii) We have

\[ f'(x) = x^4 - 2x^2 + 1 = (x^2 - 1)^2. \]

Since \( f \) is a polynomial, it is continuous on \([1, \pi] \) and differentiable on \((1, \pi) \). By the Mean Value Theorem, there exists some \( c \in (1, \pi) \) such that \( f'(c) = \frac{f(\pi) - f(1)}{\pi-1} \). But \( f'(c) = (c^2 - 1)^2 > 0 \) since \( c > 1 \) (it’s important that \((c^2 - 1)^2 > 0\) is a strict inequality). Thus \( \frac{f(\pi) - f(1)}{\pi-1} > 0 \) and \( f(\pi) - f(1) > 0 \) and \( f(\pi) > f(1) \).

\[ \square \]

Problem 6. Find \( \lim_{x \to 0^+} x \ln x \).

Solution. Rewrite the limit as \( \lim_{x \to 0^+} \frac{\ln x}{1/x} \). As \( x \to 0^+ \), we have \( \ln x \to -\infty \) and \( 1/x \to \infty \) so \( \lim_{x \to 0^+} \frac{\ln x}{1/x} \) is an indeterminate form. Thus we have

\[ \lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \text{ by L’Hospital’s Rule} = \lim_{x \to 0^+} -x = 0. \]

\[ \square \]
Problem 7. A pitcher at \( P = (0, 2) \) throws a ball whose position at time \( t \) is \( B(t) = (x(t), y(t)) \) where \( x(t) = t \) and \( y(t) = -\frac{1}{32}(x(t))^2 + 2 \). You are watching the ball from the point \( A = (8, 2) \). Let \( \theta(t) \) be the angle \( \angle PAB(t) \). How fast (in radians) is \( \theta(t) \) changing when the ball hits home plate, i.e. when \( B(t) = (8, 0) \)?

Solution. We have to express \( \theta(t) \) in terms of \( x(t) \) and \( y(t) \) somehow. Consider the following diagram:

\[
\cot(\theta(t)) = \frac{8 - x(t)}{2 - y(t)} = \frac{8 - t}{2 - \left(-\frac{1}{32}t^2 + 2\right)} = \frac{32(8 - t)}{t^2}.
\]

We have \( B(t) = (8, 0) \) if \( t = 8 \). Differentiating with respect to \( t \) gives

\[
-csc^2(\theta(t))\theta'(t) = \frac{32(-1)(t^2) - 32(8 - t)(2t)}{(t^2)^2}
\]

and at \( t = 8 \) we have \( \theta(8) = \frac{\pi}{2} \) and \( \csc\left(\frac{\pi}{2}\right) = 1 \) so

\[
-csc^2(\theta(8))\theta'(8) = \frac{32(-1)(8^2) - 32(8 - 8)(2 \cdot 8)}{(8^2)^2}
\]

implies

\[
\theta'(8) = \frac{1}{2}.
\]

□

Problem 8. Consider the graph of the function \( f(x) = x^3 - x \).

(a) Find the regions of the graph where \( f(x) \) is increasing. Find the regions of the graph where \( f(x) \) is decreasing. Find all local extrema of \( f \). Which of these are global extrema?

(b) Find the regions of the graph where \( f(x) \) is concave up. Find the regions of the graph where \( f(x) \) is concave down. Find all inflection points.

Solution. (a) We have

\[
f'(x) = 3x^2 - 1.
\]
Thus $f'(x) > 0$ on $(-\infty, -\frac{1}{\sqrt{3}}) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$ and $f'(x) < 0$ on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. So $f$ is increasing on $(-\infty, -\frac{1}{\sqrt{3}})$, decreasing on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, and increasing on $\left(\frac{1}{\sqrt{3}}, \infty\right)$. The critical points of $f$ are at $x = \pm \frac{1}{\sqrt{3}}$. The point $\left(-\frac{1}{\sqrt{3}}, f\left(-\frac{1}{\sqrt{3}}\right)\right)$ is a local maximum and $\left(\frac{1}{\sqrt{3}}, f\left(\frac{1}{\sqrt{3}}\right)\right)$ is a local minimum by the First Derivative Test, since $f'$ changes from positive to negative at $x = -\frac{1}{\sqrt{3}}$ and from negative to positive at $x = \frac{1}{\sqrt{3}}$.

(b) We have

$$f''(x) = 6x$$

so $f$ is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$. The unique inflection point is at $x = 0$.

Figure 1. Graph of $f(x) = x^3 - x$