Problem 1. (i) State Rolle’s Theorem.
(ii) State the Mean Value Theorem.
(iii) Prove the Mean Value Theorem. You may assume Rolle’s Theorem.

Solution. (i) Let \( f \) be a function continuous on \( [a, b] \) and differentiable on \( (a, b) \). Suppose \( f(a) = f(b) \).
Then there exists \( c \in (a, b) \) such that \( f'(c) = 0 \).

(ii) Let \( f \) be a function continuous on \( [a, b] \) and differentiable on \( (a, b) \). Then there exists \( c \in (a, b) \) such that
\[
\frac{f(b) - f(a)}{b - a} = f'(c).
\]

(iii) Let \( f \) be a real-valued function that is continuous on \( [a, b] \) and differentiable on \( (a, b) \). Define
\[
g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a} (x - a).
\]
Note that \( g \) is a real-valued function that is continuous on \( [a, b] \) and differentiable on \( (a, b) \). Note that \( g(a) = g(b) \). Hence, by Rolle’s Theorem, there exists \( c \in (a, b) \) such that \( g'(c) = 0 \). But \( g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \). Hence \( f'(c) = \frac{f(b) - f(a)}{b - a} \).
\[
\square
\]

Problem 2. In each of the following cases, evaluate \( \frac{dy}{dx} \).

(i) \( y = \sin(x) \ln |x| \).
(ii) \( y = \frac{x^2}{\cos x} \)
(iii) \( y = (10x^4 - 5)^{20} \)
(iv) \( y = \sin(x \sin x) \)
(v) \( y = \tan(x^2)e^{3x} \)
(vi) \( y^4 - 3y = e^{2x} \) (you should leave your answer in terms of \( y \) and \( x \))

Solution. (i) Use the product rule:
\[
\frac{dy}{dx} = \cos(x) \ln |x| + \sin(x) \frac{1}{x}.
\]

(A common mistake was to say that \( \frac{d}{dx} (\ln |x|) = \frac{1}{|x|} \) instead of \( \frac{1}{x} \). Notice that the domain of definition for the function \( \sin(x) \ln |x| \) is \( (-\infty, 0) \cup (0, \infty) \), so \( \cos(x) \ln |x| + \sin(x) \frac{1}{x} \) and \( \cos(x) \ln |x| + \sin(x) \frac{1}{x^2} \) are actually different functions on \( (-\infty, 0) \cup (0, \infty) \). A sanity check that you can use is that the derivative of an odd function should be even, and vice versa.)

(ii) Use the quotient rule:
\[
\frac{dy}{dx} = \frac{(\cos(x))(2x) - (-\sin(x))(x^2)}{(\cos x)^2}.
\]

(iii) Use the chain rule and the power rule:
\[
\frac{dy}{dx} = 20(10x^4 - 5)^{19}(40x^3).
\]

(iv) Use the chain rule and the product rule:
\[
\frac{dy}{dx} = \cos(x \sin x) \cdot (\sin x + x \cos x)
\]
(v) Use the product and chain rules:
\[ \frac{dy}{dx} = \sec^2(x^2)(2x)e^{x^3} + \tan(x^2)e^{x^3}(3x^2) \]

(vi) Differentiate implicitly:
\[ 4y^3 \frac{dy}{dx} x + y^4 - 3 \frac{dy}{dx} = 2e^{2x} \tag{1} \]
which implies
\[ \frac{dy}{dx} \cdot (4y^3x - 3) = 2e^{2x} - y^4 \]
so
\[ \frac{dy}{dx} = \frac{2e^{2x} - y^4}{4y^3x - 3} \]

(A common mistake was to forget the “x” hiding behind the \( \frac{dy}{dx} \) in the first term of \( (1) \), so that the final answer became \( \frac{2e^{2x} - y^4}{4y^3 - 3} \) instead.)

\[ \square \]

**Problem 3.** Showing your working carefully, calculate \( \frac{dy}{dx} \) when \( y = x^e^x \). Give your answer in terms of just \( x \).

**Solution.** We have \( y = x^e^x \). Then \( \ln(y(x)) = \ln(x^e^x) = e^x \ln x \). Differentiating both sides with respect to \( x \) gives \( \frac{1}{y} \frac{dy}{dx} = e^x \ln x + e^x \frac{1}{x} \). Thus
\[ \frac{dy}{dx} = x^e^x \left( e^x \ln x + e^x \frac{1}{x} \right) \]

\[ \square \]

**Problem 4.** I start with 5 ties to wear on special occasions. My hair-dresser tells me that my separation anxiety is because I don’t have enough nice clothes. Taking the advice to heart, I start shopping and my tie collection starts to grow exponentially. After 3 days I have 12 ties. How long will it be before I have 100 ties? You do not need to simplify or evaluate your answer.

**Solution.** We have \( T(t) = T(0)e^{kt} \) where \( k \) is constant. We know \( T(0) = 3 \). So \( T(t) = 3e^{kt} \). Also, \( T(3) = 12 \). Hence \( 12 = T(3) = 3e^{k3} \), which implies \( k = \frac{1}{3} \ln 4 \). So \( T(t) = 3e^{\frac{t}{3} \ln 4} \). Let \( t_0 \) be the time at which I have 100 ties, i.e. \( T(t_0) = 100 \). Then \( 100 = T(t_0) = 3e^{\frac{t_0}{3} \ln 4} \). Solving for \( t_0 \) gives \( t_0 = \frac{3\ln(\frac{100}{3})}{\ln 4} \) days.

\[ \square \]

**Problem 5.** Which point on the graph of \( y = \sqrt{x} \) is closest to the point \( (4,0) \)?

**Solution.** Let \( D(x) \) be the distance between \( (4,0) \) and \( (x, \sqrt{x}) \). We have \( (D(x))^2 = (x-4)^2 + (\sqrt{x} - 0)^2 = (x-4)^2 + x \). Differentiating with respect to \( x \) gives
\[ 2D(x)D'(x) = 2(x-4) + 1 \]
so
\[ D'(x) = \frac{2(x-4) + 1}{\sqrt{(x-4)^2 + x}} \]
Suppose that $D(x)$ takes a minimum at $x = x_0$. Then $D'(x_0) = 0$. This requires $2(x_0 - 4) + 1 = 0$, or $x_0 = \frac{7}{2}$. We have

$$D''(x) = \frac{2\sqrt{(x-4)^2 + x} - (2x-4) + 1}{2\sqrt{(x-4)^2 + x}} (2x-4) + 1$$

which is positive at $x_0 = \frac{7}{2}$, so by the Second Derivative Test (page 295) we have that $D(x_0)$ is a local minimum. It’s a global minimum since $D'(x) > 0$ on $(\frac{7}{2}, \infty)$ and $D'(x) < 0$ on $(0, \frac{7}{2})$. So the closest point is $(\frac{7}{2}, \sqrt{\frac{7}{2}})$.

\[\square\]

Problem 6. A salt crystal is growing in a super-saturated solution of salt. It is a perfect cube and its length, width and height are all growing at a rate of 1 mm per day. What is the rate of increase of the volume of the cube when its length, width and height all equal 10 mm?

Solution. Let $V(t)$ and $w(t)$ be the volume and width of the cube at time $t$, respectively. We have $V(t) = (w(t))^3$. So $V'(t) = 3(w(t))^2w'(t)$, and $w'(t) = 1$ for all $t$. So when $w(t) = 10$, we have $V'(t) = 3(10^2)(1) = 300$ mm$^3$/day.

\[\square\]

Problem 7. Showing your working carefully, evaluate

$$\lim_{x \to 0} \frac{\sin^2(x)}{4x^2}.$$ 

If you use a rule to change the limit, then you should name that rule each time you use it.

Solution. We have

$$\lim_{x \to 0} \frac{\sin^2(x)}{4x^2} = \lim_{x \to 0} \frac{2\sin(x)\cos(x)}{8x} \quad \text{by L’Hospital’s Rule}$$

$$= \lim_{x \to 0} \frac{2\cos^2(x) - 2\sin^2(x)}{8} \quad \text{by L’Hospital’s Rule}$$

$$= \frac{1}{4}.$$ 

\[\square\]

Problem 8. What is $9^{\log_{49} 7}$?

Solution. We have $\log_{49} 7 = \frac{\ln 7}{\ln 49} = \frac{\ln 7}{2 \ln 7} = \frac{1}{2}$ so $9^{\log_{49} 7} = 9^{1/2} = \pm 3$. 

\[\square\]