Overview of Scholze's work and related works

F/Qp, GLn(F), Dx inv 1/n as before

Have - the Lubin-Tate tower (MLT,k) k ∈ GLn(F) / F
- the Gross-Hopkins period map

\[ \pi_{GH} : MLT,k \to IP := IP_{F}^{\nu} \] 

GLn(F) x Dx x Weil equiv

IP as the Bruhat-Serre ri variety for Dx/F

Main Thm (Scholze, rough form)

∃ natural exact functor

( sm adm Fp-rep of GLn(F)) \to (Dx x Weil equiv sheaves on IP_{ét})

\[ \pi \mapsto \Omega F \pi \]

s.t. (1) for C ⊂ F complete alg closed

- Hi(P_C, F \pi) is an adm rep of Dx,
- Wp-action extends contly to a GilP-action
- it is indep of choice if C
- it vanishes when i > 2(n-1)

(2) GilP(Qp) : satisfy a local-global compatibility w/ cohom of Shimura curve

(3) compatible with CEGGSPS & Knight (later)

Upshot: Gives a geometric construction of candidates

of mod p / p-adic Langlands & Jacquet-Langlands
Pem. (1) indep of C: non-trivial

\[ k = \bar{k}/\mathbb{F}_p, \quad H'(A_k, \mathbb{F}_p) \text{ depends on } k \ (\text{Arnn-Schreer}) \]

(2) expected to hold for $GL_n(F)$ (ongoing thesis work?)

Test of talk: explain background, related works and a project.
- start discussion on Scholze's paper.

§. Background etc

Recall:

Local Langlands correspondence for $GL_n(F)$

\[ G \cong \text{(n-dim Frob s.s. Weil-Deligne reps of } F) \]

\[ \leftrightarrow (\text{irred sm. adm. reps of } GL_n(F)) \]

satisfying nice properties.

Local Jacquet-Langlands correspondence for $GL_n(F)$ & $D^x$

\[ G \cong (\text{irred disc. series reps of } GL_n(F)) \]

\[ \leftrightarrow (\text{irred sm. adm. reps of } D^x) \]

satisfying some char reln.
Q: Can we upgrade LLC & LJL using $p$-adic topology?

$GL_2(\mathbb{Q}_p)$

Thin (Colmez + Breuil, Berger, Kisin, Emerton, Dospinescu, Parkunas, ...)

$\equiv$ (abs irred unitary adn Banach rep of $GL_2(\mathbb{Q}_p)$)

- non-ordinary

$\leftarrow$ (abs irred 2-dim $p$-adic rep of $GL_2(\mathbb{Q}_p)$)

s.t. (1) compatible with mod $p$ Langlands by Breuil

(2) compatible with $p$-adic families (deformations)

(3) encodes LLC (via taking loc. alg. vectors)

(4) satisfies local-global compatibility

Rem.: Colmez uses $(4, 17)$-module theory

L0 works well only for $GL_2(\mathbb{Q}_p)$

mysterious beyond $GL_2(\mathbb{Q}_p)$ + lots of pathologies.
Scholze's work: \((\mod p \text{ rep of } \text{GL}_n(F)) \rightarrow (\mod p \text{ rep of } \mathbb{D}^* \times \text{GL}_n)\).

related works
- Caraiani - Emerton - Gee - Geraghty - Paskunas - Shin
  \((\text{mod dim rep of } \text{GL}_n) \rightarrow (\text{rep of } \text{GL}_n(F))\)
  using global method (patching)

Rem: Scholze studies: the relation.
- Colmez - Dospinescu - Nizioł computed.
  - cohom of the 1-dim Drinfeld tower & proved it encodes pLC for \(\text{GL}_2(\mathbb{F}_p)\)
  - cohom of the mod dim Drinfeld half-space
- Ludwig, Johansson - Ludwig, Paskunas: further study of Scholze's functor
- Knight: \(p\)-adic JL using Drinfeld tower + global method
  \& \(\text{GL}_1\)
  \& Knight-Chojnacki, Howe: further study of \(p\)-adic JL.

A project: Slogan: think positive because we are equal!

\(F = \mathbb{F}_q(\!(t)\!))\)

Fact: \(\text{LLC} \& \text{LJL exist!}\)

Q: How about \(\text{mod } t \slash t\)-adic Langlands \& JL?

Good news:
\((\text{MHT, } k) \& \Pi_{CM}\) exist in equal char!

Cal Foscolo \& Calabrese (w.p.) \subset \text{Calabrese} much smaller

Step 0: Learn these players

Step 1: Redo Scholze in this setting

Step 2: Find something interesting.

one possibility: compare Scholze's & ours

+ use \( \text{mod } p \) Kazhdan \& hom (if it exists)?
  + directly use geometry
5. Construction of $\mathcal{O}_\pi$

Recall: Scholze–Weinstein proved

$\exists$ preperfection $\mathcal{M}_{LT,\infty} \sim \varprojlim_k \mathcal{M}_{LT,k}$

$\text{homeo} \sim \text{"$l^\infty$} \text{Ai} \to \text{A dense image"}$

$P = p_{\mathbb{F}_p}^{\infty} \mathcal{G}_{\mathbb{Q}}(F) \times D^x \text{ cont}$

Recall: $G \circ X$ is cont if $\exists X = U$ $Sp_{\mathbb{A}^n}(A_i, A_i')$, $\exists H_i \subset G$

loc profinite adele $sp$

s.t. $H_i$-stability $\Rightarrow H_i \times A_i \to A_i$ cont

$\text{e.g.} \mathcal{G}_{\mathbb{Q}}(\mathbb{A}_p) \cong \mathbb{P}_{\mathbb{Q}}^1$ \hspace{1cm} $1 + p \mathbb{M}_2(\mathbb{Z}_p) \cong \{ x | x \in \mathbb{F}_p \}$

Def.: Define equiv étale site $(P/P^x)$ ét by

$\text{obj} = (D^x \times U, U \to P \text{ D}^x \text{ eq étale})$

$\text{mor} = U \to U'$ $\text{G eq étale}$

$\text{cov} = \{ U_i \to U \text{} \text{ s.t. } 1 \notin U \cap f_i(1U_i) \}$

Prop. $\pi$ adm $\mathbb{F}_p$-rep of $\mathcal{G}_{\mathbb{Q}}(F)$

Define $\mathcal{O}_{\pi}$ by $(U \to P) \mapsto \text{Map}_{\text{cont}, \mathcal{G}_{\mathbb{Q}}(F) \times D^x}(1U \times \mathcal{M}_{LT,\infty}, 1, \pi)$

Then $\mathcal{O}_{\pi}$ is a Weil-ét sheaf on $(P/P^x)$ ét

Moreover

1. $\pi \mapsto \mathcal{O}_{\pi}$ exact

2. $\forall \bar{x} = Sp_{\mathbb{A}^n}(C, C^+)$ $\to P$ geom, pt

$\mathcal{O}_{\pi, \bar{x}} \cong \pi$

$\Rightarrow (P) \bar{u} \to (P/P^x \bar{u})$ pull back
Sketch rigorous proof requires general discussions on \((X/\mathcal{E})_\et\)

\(\mathcal{F}_\pi\): sheaf follows from gluing cont maps \(|U_i \times \mathcal{MLT},\omega| \to \pi\)

\(\text{key: } \delta:\{U_i\} \subseteq \{U\} \text{ open}\)

\(\text{description of stalk}\)

choose a lift \(\tilde{x} \neq \tilde{x} \to \mathcal{MLT},\omega\)

\(\downarrow \mathcal{IP}\)

Fix a cofiltered inv system \(\mathcal{F}\) can affinoid etal nbds \(\tilde{x} \neq \tilde{x}\) refining \((\mathcal{MLT},\mathcal{K})_k\)

\[
\text{Fact } \quad \mathcal{F}_{\mathcal{F}_\pi, \tilde{x}} = \lim_{\longrightarrow} \mathcal{F}(\mathcal{MLT},\omega, \mathcal{P}) \text{ cont, Glun(F)} (|U_i \times \mathcal{MLT},\omega|, \pi)
\]

\(\delta \downarrow \mathcal{IP}\)

\(\pi\)

Define \(\delta(F) = \mathcal{F}((\tilde{x}, \tilde{x}))\)

\(\text{inj: } \bigcup U_i \text{ conn } \to \bigcup U_i \times \mathcal{MLT},\omega \text{ s Glun(F) }\)

\(\text{transitive}\)

\(\text{surj: } f \in \mathcal{F} \Rightarrow f \text{ is inv under } K_\mathcal{M} = \bigcup \mathcal{M}_n(\mathcal{O}) \text{ for } \mathcal{M}\)

\(\text{choose } U_i \text{ s.t. } U_i \to \mathcal{MLT},\mathcal{K}\)

\(\downarrow \mathcal{IP}\)

Since \(\mathcal{MLT},\mathcal{K}_\mathcal{M} : \text{surj with fibers } = \text{Glun(F) }/K_\mathcal{M}\)

\[
\mathcal{F} : |U_i \times \mathcal{MLT},\omega| \to |U_i \times \mathcal{MLT},\mathcal{K}_\mathcal{M}| \to \text{Glun(F) }/K_\mathcal{M} \to \pi
\]

\(\mathcal{Glun(F)} \text{ - equiv}\)

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