Constancy of generalized Hodge-Tate weights of a $p$-adic local system

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Galois Representations – Examples

Goal

Study a family of Galois representations parametrized by a variety $\mathbb{Q}_p$.

Let $k$ be a finite extension of $\mathbb{Q}_p$. A Galois representation is a continuous group homomorphism

$$\rho: \text{Gal}_k := \text{Gal}(\overline{k}/k) \longrightarrow \text{GL}_r(\mathbb{Q}_p).$$

- $p$-adic cyclotomic character

$$\chi: \text{Gal}_k \to \text{Gal}(k(\mu_{p^\infty})/k) \hookrightarrow \mathbb{Z}_p^\times$$

- Tate module of an elliptic curve $E/k$:

$$T_pE := \lim_{\leftarrow m} E[p^m](\overline{k}) \cong \mathbb{Z}_p^2$$

- étale cohomology of an algebraic variety $Y$ over $k$:

$$H^n_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_p)$$
Theorem (Tsuji, Faltings, Nizioł,...)

\[ H^n_{\text{ét}}(Y_k, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \cong H^n_{\text{dR}}(Y/k) \otimes_k B_{\text{dR}}. \]

Here \( B_{\text{dR}} \) is \( p \)-adic analogue of \( \mathbb{C} \) (de Rham period ring).

This implies:

\[ H^n_{\text{ét}}(Y_k, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \cong B_{\text{dR}}^{\text{dim} H^n_{\text{dR}}} \text{ (trivial Gal}_k\text{-representation}/B_{\text{dR}}\text{)} \]

Such a Galois representation is called de Rham.

Hierarchy in Galois representations:

(general Galois representations)

\[ \bigcup \]

(de Rham representations)

\[ \bigcup \]

(\( \rho \) coming from étale cohomology)
Generalized Hodge-Tate weights

There are a lot of non de Rham representations:

\[ \chi^\alpha : \text{Gal}_k \longrightarrow \mathbb{Z}_p^\times \text{ for each } \alpha \in \mathbb{Z}_p. \]

**Fact**

\( \chi^\alpha \) is de Rham if and only if \( \alpha \in \mathbb{Z} \).

**Question**

How to study a general Galois representation \( \rho \)?

Sen associated **generalized Hodge-Tate weights** \( \alpha_1, \ldots, \alpha_{\dim \rho} \in \overline{k} \) to \( \rho \).

- \( \chi^\alpha \sim \alpha \)
- \( T_pE \sim 0, 1 \)
Main Question

How about a family of Galois representations parametrized by a variety?

Let $X$ be an algebraic variety over $k$ and consider a continuous representation

$$\rho_X : \pi_1^{\text{ét}}(X) \to \text{GL}_r(\mathbb{Q}_p).$$

Then for each (closed) point $x \in X$, we have

$$\text{Gal}_{k(x)} = \pi_1^{\text{ét}}(\text{Spec } k(x)) \to \pi_1^{\text{ét}}(X) \xrightarrow{\rho_X} \text{GL}_r(\mathbb{Q}_p).$$

This means $\rho_X$ is a geometric family of Galois representations!

- $\pi_1^{\text{ét}}(X)$-representation $\rho_X \leftrightarrow \mathbb{Q}_p$-local system on $X$ (étale sheaf)
- More generally, we work on a rigid analytic variety $X$ over $k$ and $\mathbb{Q}_p$-local systems on $X$. 
Main Result

Set-up:
- $X$: a smooth rigid analytic variety over $k$ (+ geometrically connected)
- $\mathbb{L}$: a $\mathbb{Q}_p$-local system on $X$

Theorem (R. Liu - X. Zhu)

*If $\mathbb{L}$ is de Rham at one point, then it is so at any point.*

Main Theorem (S.)

Generalized Hodge-Tate weights of $\mathbb{L}$ are constant on $X$.

Key ingredients of Main Theorem:
(a) Sen’s endomorphism
(b) Geometric $p$-adic Riemann-Hilbert correspondence
(a) Sen’s endomorphism

\[ k_\infty := k(\mu_{p\infty}), \quad K := \widehat{k_\infty} \]

**Sen’s Theory**

Sen constructed a functor

\[ \text{Gal}_k \text{-representation } V \sim (\mathcal{H}(V), \phi_V). \]

- \( \mathcal{H}(V) \) is a \( K \)-vector space of \( K \)-dimension \( \dim_{\mathbb{Q}_p} V \).
- \( \phi_V \in \text{End}_K \mathcal{H}(V) \) (Sen’s endomorphism).

**Definition:** generalized Hodge-Tate weights are the eigenvalues of \( \phi_V \).

**Remark:**

- \( \mathcal{H}(V) := (V \otimes_{\mathbb{Q}_p} \widehat{k})^{\text{Gal}(\overline{k}/k_\infty)}. \)
- \( \phi_V \) comes from \( \text{Gal}(k_\infty/k) \)-action on \( \mathcal{H}(V) \).
Toward the constancy of generalized Hodge-Tate weights

To prove Main Theorem...

- Define $\mathcal{H}(L)$ and $\phi_L$ for a $\mathbb{Q}_p$-local system $L$ on $X$.
- Prove the eigenvalues of $\phi_L$ are constant.

Natural guesses:

- $\mathcal{H}(L)$ should be a vector bundle on $X_K$ and $\phi_L \in \text{End}_{X_K} \mathcal{H}(L)$.
- To prove constancy, need more geometric inputs.
  (e.g. derivatives w.r.t. coordinates are zero.)


Riemann-Hilbert correspondence/$\mathbb{C}$

$X$: complex manifold

$\mathbb{C}$-local systems $L$ on $X$ ("solutions") $\leftrightarrow$ vector bundles with integrable connections $(\mathcal{E}, \nabla: \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X)$ on $X$ ("differential equations")

Our case will be: $L$ on $X \sim (\mathcal{E}, \nabla)$ on $X \hat{} \otimes K[[t]]$
There is a natural functor

\[ \mathbb{Q}_p\text{-local system } L \text{ on } X \rightsquigarrow (E_L, \nabla). \]

- \( E_L \) is a vector bundle on \( X \hat{\otimes} K[[t]] \).
- \( \nabla \) is an integrable connection on \( X \hat{\otimes} K((t)) \) and it has log poles along the divisor \( X_K \).

Moreover, if we set

\[ H(L) := E_L|_{X_K}, \quad \text{and} \quad \phi_L := \text{Res}_{X_K} \nabla \in \text{End } H(L), \]

then \((H(L), \phi_L)\) is a natural generalization of \((H(V), \phi_V)\).

- \( E_L := \nu_* (L \otimes O_{\mathbb{B}^+_{dR}}) \otimes \text{Gal}(k_\infty/k) \) \((\nu : X_{K, \text{pro\acute{e}t}} \to X_{K, \text{\acute{e}t}})\).
- \( \nu_* O_{\mathbb{B}_{dR}} = O_X \hat{\otimes} \mathbb{B}_{dR}(K) \approx O_X \hat{\otimes} K((t)). \)
- Analysis of this connection \( \rightsquigarrow \) constancy of eigenvalues of \( \phi_L \) on \( X_K \).