Math 55, Probability Worksheet Solutions
March 17, 2014

1. What is the probability that a positive integer less than or equal to 100 selected at random is divisible by either 5 or 7?

   Solution: Let E be the event described. Then $p(E) = \frac{|E|}{|S|} = \frac{a_5+a_7-a_{35}}{100}$ by the inclusion-exclusion principle where a_i is the number of positive integers less than or equal to 100 and divisible by i. One out of every i consecutive integers is divisible by i so $a_i = \lfloor 100/i \rfloor$, so $a_5 = 20, a_7 = 14$, and $a_{35} = 2$. So $p(E) = \frac{20+14-2}{100} = \frac{32}{100} = \frac{8}{25}$.

2. Which is more likely, getting a sum of 9 from rolling 2 dice, or getting a sum of 9 from rolling 3 dice?

   Solution: There are 6^2 possible outcomes from rolling 2 dice and only 4 ordered pairs of integers less than or equal to six that sum to 9, namely $(3,6),(4,5),(5,4)$, and $(6,3)$. So the probability of getting a sum of 9 from rolling 2 dice is $\frac{4}{36} = \frac{1}{9}$.

   There are 6^3 possible outcomes from rolling 3 dice. The following is a complete list of unordered triplets of integers less than or equal to six that sum to 9:

$$\begin{gathered}(6,2,1)\\(5,3,1)\\(5,2,2)\\(4,3,2)\\(4,4,1)\\(3,3,3)\end{gathered}$$

   The first, second, and fourth of these have $3! = 6$ possible orderings, the third and fifth have 3 possible orderings and the last has 1 possible ordering. So there are $3 \cdot 6 + 2 \cdot 3 + 1 = 25$ outcomes that sum to 9. So the probability of getting a sum of 9 from rolling 3 dice is $\frac{25}{216}$.

   Since $\frac{25}{216} > \frac{1}{9}$, it is more likely to sum to 9 when rolling 3 dice.

3. Let's say you place 8 rooks randomly on a chess board (so that they all sit in different squares). What is the probability that they will not be able to attack each other (i.e. no two will be in the same row or column)?

   Solution: There are 64 squares on the chessboard, so the total number of ways to place two rooks in any two distinct squares is $\binom{64}{2} = \frac{64 \cdot 63}{2}$.

   Now we count the number of ways to place two rooks that are not in the same row or column. There are 64 ways to place the first rook. Now there are 7 remaining squares in that row and 7 remaining squares in that column where the second rook cannot be placed, as well as the already taken square. So there are $64 - 2 \cdot 7 - 1 = 49$ ways to place the remaining rook. So there are $64 \cdot 49$ ways to place 2 rooks on a chessboard if order matters. Order doesn't matter in our case, so we divide by $2! = 2$ (in other words, $64 \cdot 49$ double counts every configuration, since we could have put down our rooks in the opposite order). So the probability that the rooks can't attack one another is $\frac{64 \cdot 49/2}{64 \cdot 63/2} = \frac{49}{63} = \frac{7}{9}$.

4. Imagine you roll a pair of dice 24 times in a row. Let $P((m,n))$ for $1 \leq m, n \leq 6$ denote the probability that the pair of integers (m,n) are rolled at least once in those 24 times.

   - Guess whether $P((6,6)) > 1/2$, $P((6,6)) < 1/2$, or $P((6,6)) = 1/2$. Don't think too hard.

   - Now compute $P((6,6))$ and see if your guess was right

     Solution: The probability that the pair $(6,6)$ is rolled on any given toss is $\frac{1}{36}$, so the probability that $(6,6)$ is NOT rolled is $\frac{35}{36}$. Each roll of a pair of dice is independent from previous and subsequent rolls, so the probability that $(6,6)$ is not ever rolled in 24 tosses is $(\frac{35}{36})^{24}$. So the probability that $(6,6)$ is rolled at least once is $1 - (\frac{35}{36})^{24} = 0.491 \cdots < 1/2$. So $P((6,6)) < 1/2$ (just barely!).

   - True or false: the probability of rolling double of any integer (i.e. rolling $(1,1), (2,2), (3,3), (4,4), (5,5)$, or $(6,6)$) at least once during 24 dice rolls is $6 \cdot P(6,6)$?

     Solution: False. The probability of NOT rolling doubles is $\frac{30}{36}$ so by the above reasoning, the probability of rolling doubles at least once is $1 - (\frac{30}{36})^{24}$. You can check computationally that $1 - (\frac{30}{36})^{24} \neq 6(1 - (\frac{35}{36})^{24})$

5. What if there were four doors in the Monty Hall puzzle, and the host still opened one empty door after your choice? What is the probability of winning if you stick to your original choice, and what is the probability of winning if you switch?

   Solution: Before the host asked if you want to switch, there was a 3/4 chance that the prize hid behind the 3 remaining doors and a 1/4 chance that the prize hid behind the door you chose. If you do not switch, the probability that you win remains 1/4. If you do switch, you have 2 options to switch to. Each has probably 1/2 of hiding the prize if the prize is indeed behind one of them (i.e. is not behind the one you chose), which we said happens with probability 3/4. So the probability that the prize hides behind each of the remaining 2 doors is $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$.

6. Imagine that you want to adopt a cat from a pet shelter. There are n cats that need homes, and they are ordered randomly. You only have room in your apartment to adopt one cat. The way this shelter works is as follows: you sit in a room and the first cat is led in. You have five minutes to play with it and then you have to decide on the spot if you want to reject it or adopt it. If you adopt it, then you go home right then without seeing the rest of the cats. If you reject it, then you reject it forever and can no longer adopt it. You continue this until you adopt a cat (or if you still haven't decided at the end then you must adopt the last cat you see). Assume that if you were able to see all the cats at once, you would be able within five minutes to order them unambiguously from 1 to n, 1 being your favorite and n being your least favorite. Thus each time you see a new cat, you can rank it in terms of all the cats you've seen before (but you have no idea about how the cats you haven't seen yet compare). What is the optimal strategy if your goal is to maximize the probability of adopting your favorite cat?

   Solution: See http://en.wikipedia.org/wiki/Secretary_problem.