Math 55, Probability Worksheet #2
March 19, 2014

1. What is the probability of each of these events when we randomly select a permutation of the numbers one through four:

   - 1 precedes 4

     Solution: By symmetry, the number of permutations where 1 precedes 4 and the number of permutations where 4 precedes 1 are equal. In every permutation we must have that either 1 precedes 4 or 4 precedes 1 so $2 \cdot |E| = |S|$ so $p(E) = \frac{|E|}{|S|} = \frac{1}{2}$.
   - 1 precedes 4 and 1 precedes 3

     Solution: If 1 precedes both 3 and 4 then it must land in one of the first 2 positions. If it lands in the first position, there are $3! = 6$ options for arranging the numbers $2, 3$, and 4 and if it lands in the second then 2 must precede it and there are $2! = 2$ options for arranging the numbers 3 and 4. So there are $6 + 2 = 8$ possible outcomes in this event. There are $4! = 24$ total outcomes so $p(E) = 8/24 = 1/3$.
   - 1 precedes 4, 1 precedes 3, and 1 precedes 2

     Solution: If 1 precedes 2, 3, and 4 then it must land in the first position, and there are $3! = 6$ options for arranging the numbers $2, 3$, and 4. So there are 6 possible outcomes in this event. There are $4! = 24$ total outcomes so $p(E) = 6/24 = 1/4$.
   - 1 precedes 4 and 2 precedes 3

     Solution: By symmetry, the number of permutations where 1 precedes 4 and 2 precedes 3 is equal to the number of permutations where 1 precedes 4 and 3 precedes 2. In every permutation where 1 precedes 4 we must have that either 2 precedes 3 or 3 precedes 2 so if E is the event that 1 precedes 4 and 2 precedes 3 and F is the event that 1 precedes 4, we have $2 \cdot |E| = |F| = |S|/2$ (as computed in part one) so $p(E) = \frac{|E|}{|S|} = \frac{1}{4}$.

2. What is the smallest number of people you can choose at random to guarantee that the probability that one of them has a birthday today is at least 1/2?

   Solution: The probability that at least one of n people has a birthday today is one minus the probability that none of the the n people has a birthday today. The probability that any given person does not have a birthday today is $\frac{364}{365}$. Thus the probability that none of the n people has a birthday today is $(\frac{364}{365})^n$, and therefore the probability that at least one person out of n random people has a birthday today is $1 - (\frac{364}{365})^n$. So we want to solve $1 - (\frac{364}{365})^n \geq 1/2$ or in other words $1/2 \geq (\frac{364}{365})^n$. Taking the log base 2 of both sides, this is the same as saying

   $$-1 \geq n \log_2(\frac{364}{365})$$

   which is equivalent to

   $$-1/\log_2(\frac{364}{365}) \leq n$$

   (since $\log_2(\frac{364}{365}) < 0$). We compute that $-1/\log_2(\frac{364}{365}) = 252.65\ldots$ so we must have at least 253 people to get a probability of at least one of them having a birthday today greater than 1/2.

3. Prove that if E and F are events then $p(E \cap F) \geq p(E) + p(F) - 1$.

Solution: Let S be the space of all possible outcomes. Then

$$p(E \cap F) = \frac{|E \cap F|}{|S|} = \frac{|E| + |F| - |E \cup F|}{|S|} = p(E) + p(F) - \frac{|E \cup F|}{|S|}.$$

Since $|E \cup F| \leq |S|$ we have $\frac{|E \cup F|}{|S|} \leq 1$ so $-\frac{|E \cup F|}{|S|} \geq -1$ so $p(E \cap F) \geq p(E) + p(F) - 1$.

4. Consider a family with n children. Let E be the event that the family has at least one child of each gender and let F be the event that the family as at most one boy. For which n (if any) are E and F independent? Assume that birth order matters, so the sets {boy, girl} and {girl, boy} are different outcomes.

Solution: Recall that E and F are independent if $p(E \cap F) = p(E) \cdot p(F)$. To solve this problem, we will explicitly compute $p(E)$ and $p(F)$. Note that the number of ways we can assign genders to a family's n children is 2^n, since order matters.

For any n, there are only 2 ways that a family might not have at least one child of each gender, namely the case where all children are male and the case where all children are female. So $p(E) = \frac{2^n - 2}{2^n}$. Furthermore, there are $n+1$ ways that a family can have at most one boy, namely the n cases with exactly one boy where the boy is the youngest, second youngest, $\ldots$, second oldest, oldest, and the one case where all children are girls. So $p(F) = \frac{n+1}{2^n}$. Finally, there are n ways for a family to have at least one child of each gender and at most one boy, namely the n ways of having exactly one boy. So $p(E \cap F) = \frac{n}{2^n}$. So the problem amounts to computing for which n we have the equality

$$\frac{n}{2^n} = \frac{2^n - 2}{2^n} \cdot \frac{n+1}{2^n}$$

or equivalently

$$2^n \cdot n = (2^n - 2)(n+1) = 2^n \cdot n - 2n + 2^n - 2$$

or, in other words

$$0 = -2n + 2^n - 2$$

which can be written as

$$n + 1 = 2^{n-1}$$

Certainly for large enough n, 2^{n-1} is far bigger than $(n+1)$, and we see from the graphs of the two functions that for $n > 0$ there is only one intersection point, after which 2^{n-1} is always bigger than $(n+1)$. We need only check for very small n to find this point. We observe:

$$\begin{aligned}
&\text{for } n = 1 : 1 + 1 > 2^0 \\
&\text{for } n = 2 : 2 + 1 > 2^1 \\
&\text{for } n = 3 : 3 + 1 = 2^2 \\
&\text{for } n = 4 : 4 + 1 < 2^3
\end{aligned}$$

so events E and F are independent if and only if $n = 3$.

5. Suppose that in the world of Harry Potter, 8% of people have magical powers, 96% of magical people have parents who are magical, and 9% of nonmagical people are squibs (i.e. have parents who are magical). What is the probability that a person selected randomly from Harry Potter's world with magical parents is magical himself?

Solution: Let E be the event that a person is magical and F be the event that a person's parents are magical. Then the problem statement tells us that

$$\begin{aligned} p(E) &= 0.08 \\ p(\neg E) &= 0.92 \\ p(F|E) &= 0.96 \\ p(F|\neg E) &= 0.09 \end{aligned}$$

We want to compute $p(E|F)$, the probability that a person is magical given that he/she has magic parents. Bayes' Theorem tells us

$$p(E|F) = \frac{p(F|E)p(E)}{p(F|E)p(E) + p(F|\neg E)p(\neg E)}$$

so in our case

$$\begin{aligned} p(E|F) &= \frac{0.96 \cdot 0.08}{0.96 \cdot 0.08 + 0.09 \cdot 0.92} \\ &= 0.481\ldots \end{aligned}$$

so the probability is about 48%.

6. Prove that if $E_1, E_2, \ldots, E_n$ are events from a finite sample space then

$$p(E_1 \cup E_2 \cup \cdots \cup E_n) \leq p(E_1) + p(E_2) + \cdots + p(E_n)$$

Solution: Let S be the space of all possible outcomes. Then

$$p(E_1 \cup \cdots \cup E_n) = \frac{|E_1 \cup \cdots \cup E_n|}{|S|} \leq \frac{|E_1| + \cdots + |E_n|}{|S|} = p(E_1) + \ldots p(E_n).$$

7. Suppose that 1 in every 10,000 children are math prodigies. Say there is a test that can be given to babies at age 3 months that will determine if the baby will grow up to be a math prodigy or not (there is no such test of course, nor is there a well-defined notion of "prodigy", but let us assume for the sake of this problem). Say that 99% of math prodigies passed the test as babies and only 0.001% of people who are not math prodigies passed the test.

   - What is the probability that someone who passes the test grows up to be a math prodigy?

     Solution: Let E be the event that a baby grows up to be a math prodigy and let F be the event that a baby passes the test. The problem statement tells us that

     $$\begin{aligned} p(E) &= .0001 \\ p(\neg E) &= 0.9999 \\ p(F|E) &= .99 \\ p(F|\neg E) &= 0.001 \end{aligned}$$

     We want to compute $p(E|F)$, the probability that a person who passes the test grows up to be a math prodigy. Bayes' Theorem tells us

     $$p(E|F) = \frac{p(F|E)p(E)}{p(F|E)p(E) + p(F|\neg E)p(\neg E)}$$

so in our case

$$\begin{aligned} p(E|F) &= \frac{0.99 \cdot 0.0001}{0.99 \cdot 0.0001 + 0.001 \cdot 0.9999} \\ &= 0.090\ldots \end{aligned}$$

so the probability is about 9%.

- What is the probability that someone who doesn't pass the test doesn't grow up to be a math prodigy?

  Solution: Here we want to compute $p(\neg E|\neg F)$, the probability that a person who doesn't pass the test doesn't grow up to be a math prodigy. By replacing $\neg E$ for E and $\neg F$ for F in the statement of Bayes' Theorem, we see that

  $$p(\neg E|\neg F) = \frac{p(\neg F|\neg E)p(\neg E)}{p(\neg F|\neg E)p(\neg E) + p(\neg F|E)p(E)}$$

  Since $p(\neg F|\neg E) + p(F|\neg E) = 1$, we have

  $$\begin{aligned} p(\neg F|\neg E) &= 0.999 \\ p(\neg F|E) &= 0.01 \end{aligned}$$

  so in our case

  $$\begin{aligned} p(E|F) &= \frac{0.999 \cdot 0.9999}{0.999 \cdot 0.9999 + 0.01 \cdot 0.0001} \\ &= 0.9999989989\ldots \end{aligned}$$

  so the probability is more than 99.99%.