THE BRAUER-MANIN OBSTRUCTION

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Abstract. In this paper we present the definition of the Brauer-Manin obstruction to the existence of global points. In order to do this, we present some basic facts from the theory of Brauer groups and central simple algebras. Additionally, we compute an example of a surface for which the Brauer-Manin obstruction explains the failure of the local-global principle.

1. Introduction

Let us imagine an eighth grade math teacher standing in front of a class. He writes a polynomial on the board, say $y^2 = x^3 + 10x + 20$, and asks the students if there are rational numbers x and y satisfying this polynomial. At first glance, this may seem like a reasonable question. After all, the students should know what a polynomial is and what a rational number is. An eighth grader can certainly plug in a rational number for x and hope to find a rational number for y. But if he spends all day plugging in values for x and find no rational point, he does not have the tools to conclude with certainty that no solution exists.

As if turns out, nobody has these tools. In fact, it is not even known if it is possible to construct an algorithm to determine whether or not a polynomial equation has rational solutions. In 1900, David Hilbert proposed a list of unsolved problems to the mathematical community, the tenth of which was whether or not there existed an algorithm (i.e. a sequence of computations that terminates after a finite number of steps) to determine the existence of a solution to a finite system of polynomial equation over the integers. In modern language, this is equivalent to asking whether or not an affine scheme X of finite type over $\mathrm{Spec}(\mathbb{Z})$ has $\mathbb{Z}$-valued points. This problem was answered in 1970 in the negative by Matiyasevich and others.

A natural extension of the problem is to ask whether scheme X of finite type over any ring R has R-valued points. Let $X(R)$ denote the set of R-valued points of X, i.e. the set of morphisms $\mathrm{Spec}(R) \to X$. Then the current state of progress toward answering this question is as follows [Poo00]:

Ring R	$\exists$ an algorithm to determine if $X(R) = \emptyset$?
$\mathbb{C}$	yes
$\mathbb{R}$	yes
$\mathbb{F}_p$	yes
$\mathbb{Q}_p$	yes
$\mathbb{Q}$	?
$\mathbb{F}_p(t)$	no
$\mathbb{Z}$	no

In this paper, we will be concerned with investigating the analogue of Hilbert's tenth problem for $R = \mathbb{Q}$. Since we know it is solvable for $R = \mathbb{Q}_p$, a good place to start is to investigate the connection between $X(\mathbb{Q}_p)$ and $X(\mathbb{Q})$. Indeed, since $\mathbb{Q} \subset \mathbb{Q}_p$, the natural map $\mathrm{Spec}(\mathbb{Q}_p) \to \mathrm{Spec}(\mathbb{Q})$ gives an injection $X(\mathbb{Q}) \hookrightarrow X(\mathbb{Q}_p)$. So if we have a "global point" of X, i.e. an element of $X(\mathbb{Q})$, then we have a "local point", i.e. an element of $X(\mathbb{Q}_p)$, for each prime p.

More generally, for any global field k, let S be the set of places of k and $\mathbb{A}_k$ be the subring of $\prod_{\nu \in S} k_\nu$ consisting of infinite-tuples $(x_\nu \in k_\nu)_{\nu \in S}$ where $x_\nu \in \mathcal{O}_{k_\nu}$ for all but finitely many ν. The set of adelic points of X, denoted $X(\mathbb{A}_k)$, is the set of maps $\mathrm{Spec}(\mathbb{A}_k) \to X$. Since $\mathbb{A}_k \hookrightarrow \prod_{\nu \in S} k_\nu$, and $X(\prod_{\nu \in S} k_\nu) = \prod_{\nu \in S} X(k_\nu)$, we can intuitively think of an adelic point of X as being given by an infinite-tuple containing an element of $X(k_\nu)$ for each $\nu \in S$. In the case of $k = \mathbb{Q}$, an adelic point is given by an element of $X(\mathbb{Q}_p)$ for each p along with an element of $X(\mathbb{R})$.

The local-global principle, also called the Hasse Principle, is the statement that $X(\mathbb{A}_k) \neq \emptyset$ if and only if $X(k) \neq \emptyset$. It is not hard to show (an we will do so in Section 3) that $X(k) \hookrightarrow X(\mathbb{A}_k)$, which gives us the "if" direction. The other implication, i.e. that $X(\mathbb{A}_k) \neq \emptyset$ implies $X(k) \neq \emptyset$, does not hold in general (though it does hold for some classes of schemes, such as Brauer-Severi varieties and degree two hypersurfaces in $\mathbb{P}^n$). Of course, if the local-global principle did hold, we would have an algorithm for determining whether or not a scheme X has $\mathbb{Q}$-valued points. So we are naturally drawn to investigate obstructions to the local-global principle.

In other words, our current picture is the containment $X(k) \subset X(\mathbb{A}_k)$, where $X(\mathbb{A}_k)$ is computable and $X(k)$ is in general not. If we could construct an *obstruction*, i.e. a computable set T such that $X(k) \subset T \subset X(\mathbb{A}_k)$, we might hope to show that, even if the local-global principle fails, we get $X(k) = \emptyset$ if and only if $T = \emptyset$.

In this paper we will investigate the *Brauer-Manin obstruction*, denoted $X(\mathbb{A}_k)^{\mathrm{Br}}$, developed by Manin in 1970. The main intention of this paper will be to prove the following:

Theorem 1.1. *Let X be a scheme of finite type over a global field k. Then $X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)$.*

The Brauer-Manin obstruction is not the answer to all of our dreams in that there exist schemes X for which $X(k) = \emptyset$ while $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$. Nevertheless, in many cases we can use $X(\mathbb{A}_k)^{\mathrm{Br}}$ to explain the failure of the local-global principle. In other words, we have

Theorem 1.2. *There exists a scheme X such that $X(k) = \emptyset$, $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$, and $X(\mathbb{A}_k) \neq \emptyset$.*

In the above situation, the information $X(\mathbb{A}_k) \neq \emptyset$ does not give a conclusive answer as to whether or not $X(k) = \emptyset$, but by computing $X(\mathbb{A}_k)^{\mathrm{Br}}$ we can conclude with certainty that $X(k) = \emptyset$.

In Section 2, we review some key facts from the theory of Brauer groups. In Section 3, we define the Brauer-Manin obstruction and prove Theorem 1.1. In Section 4, we construct an example of a surface X for which $X(k) = \emptyset$, $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$, and $X(\mathbb{A}_k) \neq \emptyset$, thereby proving Theorem 1.2.

2. Brauer Groups

Let k be any field and k^s the separable closure of k. We have three equivalent definitions of the Brauer group of k, denoted $\mathrm{Br}(k)$.

Definition 1. The Brauer group of a field k is defined to be any of the following isomorphic groups:

(1) $\mathrm{H}^2_{\mathrm{et}}(\operatorname{Spec} k, \mathbb{G}_m)$
(2) $\mathrm{H}^2(G_k, (k^s)^\times)$, where $G_k := \mathrm{Gal}(k^s/k)$
(3) {Central Simple Algebras over k}/$\sim$, where "$\sim$" will be defined in Section 2.2

The bulk of this section is involved with understanding this definition. First, we will explain the equivalence of the first two definitions. Then, we will discuss central simple algebras in detail, give a group structure to (3), and sketch an equivalence between the last two definitions of $\mathrm{Br}(k)$. Finally, we will generalize the definition of $\mathrm{Br}(k)$ to define the Brauer group $\mathrm{Br}(X)$ of a scheme X of finite type over k.

2.1. **Etale and Galois Cohomology.** Let $(\operatorname{Spec} k)_{\mathrm{et}}$ denote the étale site over $\operatorname{Spec} k$ and $G_k := \mathrm{Gal}(k^s/k)$ the absolute Galois group.

Proposition 2.1. [Poo08, Theorem 6.4.5]

(1) *There is an equivalence of categories:*
$$\{\textit{sheaves of sets on } (\operatorname{Spec} k)_{\mathrm{et}}\} \leftrightarrow \{G_k \textit{ sets}\}$$
$$\mathscr{F} \mapsto \varinjlim \mathscr{F}(\operatorname{Spec} L)$$
The limit is taken over all Galois extensions L of k.
(2) *Under the equivalence from (1), the global sections functor $\mathscr{F} \mapsto \mathscr{F}(\operatorname{Spec} k)$ corresponds to the functor of invariants $S \mapsto S^{G_k}$).*

The proposition implies that $H^i_{\mathrm{et}}(\operatorname{Spec} k, \mathscr{F}) \simeq H^i(G_k, \varinjlim \mathscr{F}(\operatorname{Spec} L))$ for all i. Since $\mathbb{G}_m(\operatorname{Spec} L) = L^\times$ and $(k^s)^\times = \varinjlim L^\times = \varinjlim \mathbb{G}_m(\operatorname{Spec} L)$, we have $H^i_{\mathrm{et}}(\operatorname{Spec} k, \mathbb{G}_m) \simeq H^i(G_k, \varinjlim (k^s)^\times)$. This proves the equivalence of the first two definitions of the Brauer group of a field.

2.2. **Central Simple Algebras.** Let k be a field and A a finitely generated k-algebra. Then we say that A is *simple* if (0) is the only two-sided ideal of A, and *central* if the center of A is k. A central simple algebra over k is a finitely-generated k-algebra that is both central and simple.

Example 1. Any matrix algebra $M_n(k)$ over k is central simple.

Example 2. Any division algebra is simple, so any central division algebra is central simple. Recall that a division algebra is an algebra D such that every element has an inverse, i.e. for every $x \in D$ there exists a $y \in D$ such that $xy = yx = 1$.

Example 3. Let k be a field of characteristic $\neq 2$ and $H(a,b)$ be the k-algebra generated by i and j such that
$$H(a,b) = k \oplus ki \oplus kj \oplus kij$$
as a k-vector space and with the relations
$$i^2 = a$$
$$j^2 = b$$
$$ij = -ji$$
where $a, b \in k^\times$. Then $H(a,b)$ is called a quaternion algebra and is central simple over k.

We will need the following results in order to make sense of and work with the third definition of the Brauer group of a field. Proofs can be found in [Mil11, Chapter IV].

Proposition 2.2. [Mil11, IV, Proposition 2.3] *Let A and A' be and k-algebras and let $Z(-)$ denote "center of $-$". Then*

$$Z(A \otimes_k A') = Z(A) \otimes_k Z(A').$$

In particular, if A and A' are central k-algebras then $Z(A) = Z(A') = k$ so $Z(A \otimes A') = Z(A) \otimes Z(A') = k \otimes_k k \simeq k$, so $A \otimes_k A'$ is central.

Proposition 2.3. [Mil11, IV, Proposition 2.6] *If A and A' are simple algebras and at least one of A or A' is central then $A \otimes_k A'$ is simple.*

Combining Proposition 2.3 with Proposition 2.2, we see that the tensor product of two central simple algebras is central simple.

For any algebra A, let A^{opp} denote the algebra whose underlying vector space is that of A and with "opposite" multiplication i.e. if $\cdot$ denotes multiplication in A and $\times$ denotes multiplication in A^{opp} then $\times$ is defined by the relation $x \times y = y \cdot x$ for all $x, y \in A^{\text{opp}}$.

Proposition 2.4. [Mil11, IV, Corollary 2.9] *For any central simple algebra A, we have*

$$A \otimes A^{\text{opp}} \simeq \text{End}_k(A) \simeq M_n(k)$$

where $n = [A : k]$.

Now let us impose the following equivalence relation on the set of central simple k-algebras:

Definition 2. Let A, B be central simple k-algebras. Then we say A is similar to B, denoted $A \sim B$, if there exist positive integers n and m such that

$$A \otimes_k M_n(k) \simeq B \otimes_k M_m(k).$$

Let $[A]$ denote the equivalence class of A under this relation.

We can now impose a group structure on the set of equivalence classes under the above relation of central simple algebras over k. We define the group operation to be $\otimes_k$ and the identity to be $[M_n(k)]$. Propositions 2.3 and 2.2 show that the set is closed under $\otimes_k$, Proposition 2.4 shows that every element has an inverse, it is easy to see that $M_n(k) \otimes M_m(k) \simeq M_{mn}(k)$, and associativity is a general property of tensor product.

The equivalence of definitions (2) and (3) of $\text{Br}(k)$ is rather technical. Briefly, from each central simple algebra over k that contains a Galois extension L over k such that $[A : k] = [L : k]^2$ we construct a 2-cocycle $\text{Gal}\, L/k \times \text{Gal}\, L/k \to (k^s)^\times$. We must check that it is well defined, is indeed a 2-cocycle, and that an equivalent central simple algebra gives a 2-cocycle that differs by a 2-coboundary. The proof can be found in detail in [Mil11, Chapter 4].

Example 4. Some Brauer group of common fields:

field k	$\mathrm{Br}(k)$
algebraically closed	0
$\mathbb{F}_{p^n}$	0
$\mathbb{R}$	$\mathbb{Z}/2\mathbb{Z}$
non-archimedean local	$\mathbb{Q}/\mathbb{Z}$

2.3. **Brauer Groups of Schemes.** Now that we understand the Brauer group of a field, we would like to generalize the definition to a scheme. The first definition of $\mathrm{Br}(k)$ turns out to be the easiest to generalize.

Definition 3. Let X be a scheme. Then $\mathrm{Br}(X) := \mathrm{H}^2_{\text{et}}(X, \mathbb{G}_m)$.

It is also possible to generalize the third definition.

Definition 4. Let X be a scheme. Then an *Azumaya algebra* is a coherent $\mathcal{O}_X$-modue A such that the fiber $A \otimes_{\mathcal{O}_X} k(x)$ over each $x \in X$ is a central simple algebra over the residue field $k(x)$.

We say that two Azumaya algebras A and A' are similar if there exist locally free sheaves $\mathscr{E}$ and $\mathscr{E}'$ such that $A \otimes_{\mathcal{O}_X} \operatorname{End} \mathscr{E} \simeq A' \otimes_{\mathcal{O}_X} \operatorname{End} \mathscr{E}'$. Then the similarity classes of Azumaya algebras with the binary operation tensor product form a group. If X is regular and quasi-projective, this group is isomorphic to $\mathrm{Br}(X)$.

3. The Brauer-Manin Obstruction

Let X be a scheme of finite type over a global field k. The goal of this section is to construct a set $X(\mathbb{A}_k)^{\mathrm{Br}}$ such that $X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)$.

We begin by showing the inclusion $X(k) \subset X(\mathbb{A}_k)$. First, let us understand how to concretely define an adelic point, i.e. a map $\operatorname{Spec} \mathbb{A}_k \to X$. As noted in the introduction, $X(\mathbb{A}_k)$ is a subset of $X(\prod_{\nu \in S} k_\nu) = \prod_{\nu \in S} X(k_\nu)$. More specifically, $X(\mathbb{A}_k)$ consists of elements of $\prod_\nu X(k_\nu)$, i.e. infinite-tuples $(g_\nu)_{\nu \in S}$ of maps $g_\nu : \operatorname{Spec} k_\nu \to X$, such that all but finitely many of the g_ν "factor through" $\operatorname{Spec} \mathcal{O}_{k_\nu}$. We need to be careful about what we mean by "factor through." Let $\operatorname{Spec} A \subset X$ such that $g_\nu(\operatorname{Spec} k_\nu) \in \operatorname{Spec} A$. Then since X is of finite type over k, we have $A = k[x_1, \ldots, x_n]/(g_1, \ldots, g_r)$ for some n and r where $g_1, \ldots, g_r \in k[x_1, \ldots, x_n]$ (note that $r < \infty$ because k is a field and hence $k[x_1, \ldots, x_n]$ is noetherian). If we let T be the finite set of places ν such that $\nu(\alpha) < 0$ for some α that is a coefficient of one of the g_i and let k_T be the localization of the integers $\mathcal{O}_k$ of k by T then we get a ring $A' := k_T[x_1, \ldots, x_n]/(g_1, \ldots, g_r)$ such that $A = A' \otimes_{k_T} k$. Thus $\operatorname{Spec} A = \operatorname{Spec} A' \times_{\operatorname{Spec} k_T} \operatorname{Spec} k$, i.e. the following is a fiber product diagram:

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d] & \operatorname{Spec} A' \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} k_T \end{tikzcd}$$

and therefore

$$\operatorname{Hom}(\operatorname{Spec} k_\nu, \operatorname{Spec} A) = \operatorname{Hom}(\operatorname{Spec} k_\nu, \operatorname{Spec} k) \times \operatorname{Hom}(\operatorname{Spec} k_\nu, \operatorname{Spec} A').$$

Now by "factor through" we mean that if we consider $g_\nu : \operatorname{Spec} k_\nu \to \operatorname{Spec} A$ as a pair $(g_\nu^k, g^{A'})$ with $g_\nu^k : \operatorname{Spec} k_\nu \to k$ and $g_\nu^{A'} : \operatorname{Spec} k_\nu \to A'$, then $g^{A'}$ factors through $\operatorname{Spec} \mathcal{O}_{k_\nu}$, where $\mathcal{O}_{k_\nu} := \{x \in k \mid \nu(x) \geq 0\}$.

Now let $x \in X(k)$ so $x : \operatorname{Spec} k \to X$ and let $f_\nu : \operatorname{Spec} k_\nu \to \operatorname{Spec} k$ be the natural map coming from the injection $k \hookrightarrow k_\nu$. We claim that the map $X(k) \to X(\mathbb{A}_k)$ sending x to the infinite-tuple $(x \circ f_\nu)_{\nu \in S}$ is well-defined and injective. The main challenge is to show that $(x \circ f_\nu)_{\nu \in S} \in X(\mathbb{A}_k)$. For this, let $\operatorname{Spec} A \subset X$ be an affine open subscheme containing $x(\operatorname{Spec} k)$. Then the morphism $x : \operatorname{Spec} k \to \operatorname{Spec} A$ corresponds to a morphism of algebras $\varphi_x : A \to k$.

As above, write $A = k[x_1, \ldots, x_n]/(g_1, \ldots, g_r)$ and define $T \subset S$ and $A' = k_T[x_1, \ldots, x_n]/(g_1, \ldots, g_r)$. Now $\varphi_x(x_i) \in k$ so there are finitely many ν such that $\nu(\varphi_x(x_i)) < 0$. Let U be the set of ν such that $\nu(\varphi_x(x_i)) < 0$ for some $i \leq n$ and let $V = T \cup U$. Then if $\nu \notin V$, the morphism $\varphi_x : A' \to k$ factors through $\mathcal{O}_{k_\nu}$ and hence x factors through $\operatorname{Spec} \mathcal{O}_{k_\nu}$. Thus the image of x is indeed in $\mathbb{A}_k$.

To see that the map is injective, let $x, y \in X(k)$ such that $(x \circ f_\nu)_{\nu \in S} = (y \circ f_\nu)_{\nu \in S}$, i.e. $x \circ f_\nu = y \circ f_\nu$ for all $\nu \in S$. Equivalently, the corresponding maps on rings $j \circ \varphi_x$ and $j \circ \varphi_y$ are equal, where $j : k \hookrightarrow k_\nu$. Since j is injective, this means that $\varphi_x = \varphi_y$ so $x = y$.

We now have a well-defined injection $i : X(k) \hookrightarrow X(\mathbb{A}_k)$. What remains is to construct $X(\mathbb{A}_k)^{\mathrm{Br}}$ that lies between $X(k)$ and $X(\mathbb{A}_k)$.

Since $\operatorname{Br}(-) = \mathrm{H}^2_{\text{et}}(-, \mathbb{G}_m)$, it is a contravariant functor. For any morphism between two schemes, say $\bullet : X \to Y$, we denote by $\hat{\bullet}$ the corresponding map of Brauer groups, i.e. $\hat{\bullet} : \operatorname{Br}(Y) \to \operatorname{Br}(X)$.

Proposition 3.1. *Let $A \in \operatorname{Br}(X)$. Then we have the following commutative diagram:*

$$\begin{tikzcd}
 & X(k) \arrow[r, hook, "i"] \arrow[d, "\varphi_A"'] & X(\mathbb{A}_k) \arrow[d, "\gamma_A"'] & & \\
0 \arrow[r] & \operatorname{Br}(k) \arrow[r, "j"] & \oplus_\nu \operatorname{Br}(k_\nu) \arrow[r, "\sum \mathrm{inv}"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

where the bottom row is exact and the maps are defined as follows:

$$\begin{aligned}
i : X(k) &\to X(\mathbb{A}_k) \\
x &\mapsto (x \circ f_\nu)_{\nu \in S} \\
\gamma_A : X(\mathbb{A}_k) &\to \oplus_\nu \operatorname{Br}(k_\nu) \\
(x_\nu)_{\nu \in S} &\mapsto \oplus_\nu \hat{x}_\nu(A) \\
\varphi_A : X(k) &\to \operatorname{Br}(k) \\
x &\mapsto \hat{x}(A) \\
j : \operatorname{Br}(k) &\to \oplus_\nu \operatorname{Br}(k_\nu) \\
\mathcal{A} &\mapsto \oplus_\nu \hat{f}_\nu(\mathcal{A})
\end{aligned}$$

Proof. The exactness of the bottom row is highly nontrivial. It relies on class field theory and can be found in [Ser79]. We have already shown that i is well-defined and injective, and certainly φ_A is well-defined. Thus we have only three things to check:

(1) γ_A is well-defined

(2) j is well-defined
(3) the square commutes

Let us begin with the third statement. Let $x \in X(k)$. Going across and then down yields:

$$x \mapsto (x \circ f_\nu)_{\nu \in S} \mapsto \oplus_\nu \widehat{(x \circ f_\nu)}(A) = \oplus_\nu \hat{f}_\nu \circ \hat{x}(A)$$

while going down and then across yields

$$x \mapsto \hat{x}(A) \mapsto \oplus_\nu \hat{f}_\nu(\hat{x}(A))$$

Thus the square commutes.

Next we show that γ_A is well-defined, i.e. that given $(x_\nu)_{\nu \in S} \in X(\mathbb{A}_k)$, we have $\hat{x}_\nu(A) = 0$ for all but finitely many ν. Indeed x_ν locally factors through $\operatorname{Spec} \mathcal{O}_{k_\nu}$ for all but finitely many ν, so for all but finitely many ν we have

$$x_\nu : \operatorname{Spec} k_\nu \to \operatorname{Spec} \mathcal{O}_{k_\nu} \to \operatorname{Spec} M \subset X$$

so

$$\hat{x}_\nu : \mathrm{Br}(X) \to \mathrm{Br}(\operatorname{Spec} M) \to \mathrm{Br}(\operatorname{Spec} \mathcal{O}_{k_\nu}) \to \mathrm{Br}(k_\nu).$$

Since $\operatorname{Spec} \mathcal{O}_{k_\nu}$ is a henselian local ring with maximal ideal m, we have $\mathrm{Br}(\operatorname{Spec} \mathcal{O}_{k_\nu}) = \mathrm{Br}(\operatorname{Spec}(\mathcal{O}_{k_\nu}/m))$. Furthermore, $\mathcal{O}_{k_\nu}/m$ is a finite field so $\mathrm{Br}(\operatorname{Spec}(\mathcal{O}_{k_\nu}/m)) = 0$. Thus the image of any $A \in \mathrm{Br}(X)$ under $\hat{x}_\nu$ is trivial, so $\hat{x}_\nu(A) = 0$ for all but finitely many ν and γ_A is well-defined.

Finally, we must show that j is well-defined. For this we need the following lemma.

Lemma 3.2. *Let $A \in \mathrm{Br}(k)$ be a central simple algebra over k. Then there exists a finite set T of places and an element $\mathcal{A} \in \mathrm{Br}(k_T)$ such that $\mathcal{A} \mapsto A$ under the natural map $\mathrm{Br}(k_T) \to \mathrm{Br}(k)$. As before, k_T denotes the localization of the integers of k by the places of T, so that as a set, $k_T = \{x \in k \mid \nu(x) \geq 0 \quad \forall \nu \in S \setminus T\}$.*

Proof. As a vector space, $A \simeq k^r$ for some integer r. Choose a basis $e_1, \dots, e_r$ for A and let $s_{ij}^h \in k$ be the *structure constants* for A, i.e. $s_{ij}^h \in k$ are such that

$$e_i e_j = \sum_h s_{ij}^h e_h$$

for all $1 \leq h, i, j \leq r$. Let T be the set of $\nu \in S$ such that $\nu(s_{ij}^h) < 0$ for some i, j, h, so T is a finite set. Let $\mathcal{A}$ be the algebra over k_T given by the same structure constants, i.e. $\mathcal{A} \simeq k_T^r$ as a vector space with algebra structure given by

$$e_i e_j = \sum_h s_{ij}^h e_h$$

which is possible since $s_{ij}^h \in k_T$ for all i, j, h. Then $\mathcal{A}$ is an Azumaya algebra over k_T, and $A = \mathcal{A} \otimes_{k_T} k$ so A is the image of $\mathcal{A}$ under the map $\mathrm{Br}(k_T) \to \mathrm{Br}(k)$. $\square$

Now for all $\nu \in S \setminus T$, the natural map $\mathcal{O}_k \to \mathcal{O}_{k_\nu}$ factors through $\mathcal{O}_{k_T}$ by the universal property of localization. So we have a commutative square:

$$\begin{tikzcd} \mathcal{O}_{k_T} \arrow[r] \arrow[d] & \mathcal{O}_{k_\nu} \arrow[d] \\ k \arrow[r] & k_\nu \end{tikzcd}$$

which gives a commutative square of Brauer groups:

$$\begin{tikzcd} \mathcal{A} \arrow[d, mapsto] & \mathrm{Br}(\mathcal{O}_{k_T}) \arrow[r] \arrow[d] & \mathrm{Br}(\mathcal{O}_{k_\nu}) \arrow[d] \\ A & \mathrm{Br}(k) \arrow[r] & \mathrm{Br}(k_\nu) \end{tikzcd}$$

Thus the image of A under the map $\mathrm{Br}(k) \to \mathrm{Br}(k_\nu)$ is the image of $\mathcal{A}$ under the composition $\mathrm{Br}(\mathcal{O}_{k_T}) \to \mathrm{Br}(\mathcal{O}_{k_\nu}) \to \mathrm{Br}(k_\nu)$. But $\mathrm{Br}(\mathcal{O}_{k_\nu})$ is trivial, so the image of A is trivial in $\mathrm{Br}(k_\nu)$. Thus the image of $A \in \mathrm{Br}(k)$ in $\mathrm{Br}(k_\nu)$ is non-zero for at most finitely many ν and hence the map j is well-defined. □

Now that we have the commutative diagram

$$\begin{tikzcd} & X(k) \arrow[r, hook, "i"] \arrow[d, "\varphi_A"'] & X(\mathbb{A}_k) \arrow[d, "\gamma_A"'] & & \\ 0 \arrow[r] & \mathrm{Br}(k) \arrow[r, "j"] & \oplus_\nu \mathrm{Br}(k_\nu) \arrow[r, "\sum \mathrm{inv}"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

we are ready finally to define $X(\mathbb{A}_k)^{\mathrm{Br}}$. Let

$$X(\mathbb{A}_k)^A := \{y \in X(\mathbb{A}_k) \mid ((\sum \mathrm{inv}) \circ \gamma_A)(y) = 0\}.$$

Then $X(k) \subset X(\mathbb{A}_k)^A$ (via the inclusion i), since

$$((\sum \mathrm{inv}) \circ \gamma_A)(i(x)) = ((\sum \mathrm{inv}) \circ j)(\varphi_A(x))$$

and $(\sum \mathrm{inv}) \circ j = 0$. We define

$$X(\mathbb{A}_k)^{\mathrm{Br}} := \bigcap_{A \in \mathrm{Br}(X)} X(\mathbb{A}_k)^A.$$

We just showed that $X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}}$ and by definition $X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)$. So $X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)$.

4. Example

In this section, we give an example, constructed by Iskovskikh, of a scheme X of finite type over $\mathbb{Q}$ such that $X(\mathbb{Q}) = \emptyset$, $X(\mathbb{A}_\mathbb{Q}) \neq \emptyset$, and $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} = \emptyset$. Thus the local-global principle fails for X, but the failure is demonstrated already in $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}}$, which we are able to compute. We say in this situation that $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}}$ "explains the failure of the Hasse principle."

Let $\mathscr{E}$ be the locally free sheaf on $\mathbb{P}^1_\mathbb{Q}$ defined by $\mathscr{E} := \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2)$, and let $Y := \mathbf{P}\mathscr{E} = \mathbf{Proj}(\mathrm{Sym}^2 \mathscr{E})$. We take our scheme X to be the zero locus of

$$(1, 1, -(3b^2 - a^2)(a^2 - 2b^2))$$

in Y (here a and b give a basis for $\mathcal{O}(2)$ so $a^2, b^2 \in \mathcal{O}(2)^{\otimes 2}$). So we have

$$\begin{tikzcd} & X \arrow[d] \\ \mathbb{A}^1_\mathbb{Q} \arrow[r, hook] & \mathbb{P}^1_\mathbb{Q} \end{tikzcd}$$

Over $\mathbb{A}^1_{\mathbb{Q}}$, the sheaves $\mathcal{O}, \mathcal{O}$, and $\mathcal{O}(2)$ become trivial, i.e. all are isomorphic to $\mathcal{O}_{\mathbb{A}^1_{\mathbb{Q}}} = \mathbb{Q}[x]$. Fix such isomorphisms and let y, z, w be the generators of $\mathcal{O}, \mathcal{O}$, and $\mathcal{O}(2)$ respectively as $\mathbb{Q}[x]$-modules. Thus locally, i.e. over $\mathbb{A}^1_{\mathbb{Q}}$, X is the zero locus of $y^2 + z^2 - (3 - x^2)(x^2 - 2)w^2$ in $\operatorname{Proj}(\mathbb{Q}[x][y, x, w])$. By $\mathbb{Q}[x][y, z, w]$ we mean the ring $\mathbb{Q}[x, y, z, w]$ graded such that x has grading zero and y, z, w each have grading one. Indeed, trivializing $\mathcal{O}(2)$ corresponds to dehomogenizing its global sections and $-(3-(a/b)^2)((a/b)^2-2) \mapsto -(3-x^2)(x^2-2)$ under the isomorphism $\mathcal{O}(2)|_{\mathbb{A}^1_{\mathbb{Q}}} \simeq \mathbb{Q}[x]$ identifying a/b with x. Furthermore, notice that the image of X restricted to the open affine $U_w = \{(a_1 : a_2 : a_3) \in \operatorname{Proj}(\mathbb{Q}[x][y, x, w]) \mid a_3 \neq 0\}$ is simply

$$\operatorname{Spec}\left(\mathbb{Q}[x, y, z] \Big/ y^2 + z^2 - (3 - x^2)(x^2 - 2)\right).$$

Thus X has $\operatorname{Spec}(\mathbb{Q}[x, y, z]/y^2 + z^2 - (3 - x^2)(x^2 - 2))$ as an open subscheme. We claim that if X defined as above then $X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$. The proof of this claim is a computation using Hensel's lemma. Though we will not prove it here, we give as a warm-up a computation to show that $X(\mathbb{Q}_7) \neq \emptyset$.

Proposition 4.1. *For X defined as above, $X(\mathbb{Q}_7) \neq \emptyset$.*

Proof. Since $\operatorname{Spec}(\mathbb{Q}[x, y, z]/y^2 + z^2 - (3 - x^2)(x^2 - 2)) \hookrightarrow X$, it is enough to check that $y^2 + z^2 - (3 - x^2)(x^2 - 2)$ has a solution in $\mathbb{Q}_7$. Note that 3 is a solution for $f(x) := (3 - x^2)(x^2 - 2)$ in $\mathbb{Z}/7\mathbb{Z}$ such that $f'(3) \not\equiv 0 \mod 7$:

$$\begin{aligned} f(3) &= (3 - 9)(9 - 2) \equiv 0 \mod 7 \\ f'(3) &= -6(9 - 2) + 6(3 - 9) \equiv 6 \mod 7 \end{aligned}$$

By Hensel's lemma, 3 extends to a root a of f in $\mathbb{Q}_7$. Then $(0, 0, a)$ is a solution to $y^2 + z^2 - (3 - x^2)(x^2 - 2)$ in $\mathbb{Q}_7$. $\square$

Next, we need to show that there exists $A \in \operatorname{Br}(X)$ such that $X(\mathbb{A}_{\mathbb{Q}})^A = \emptyset$. Since $X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} := \bigcap_{A \in \operatorname{Br}(X)} X(\mathbb{A}_{\mathbb{Q}})^A$, we will have shown that $X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} = \emptyset$.

Let $K = \operatorname{Frac}(\mathbb{Q}[x, y, z]/y^2 + z^2 - (3 - x^2)(x^2 - 2))$ be the function field of X and let $A \in \operatorname{Br}(K)$ be the quaternion algebra over K defined by

$$\begin{aligned} i^2 &= 3 - x^2 \\ j^2 &= -1 \end{aligned}$$

Since $3 - x^2$ and -1 are units in K, A is a central simple algebra.

Theorem 4.2. [Poo08, Theorem 6.6.18] *If X is a regular integral noetherian scheme and $X^{(1)}$ the set of codimension 1 points then*

$$0 \to \operatorname{Br}(X) \to \operatorname{Br}(K) \to \oplus_{x \in X^{(1)}} \mathrm{H}^1_{\mathrm{et}}(k(x), \mathbb{Q}/\mathbb{Z})$$

is exact. Here $k(x)$ denotes the residue field of X at x.

Claim 4.3. *There exists $\mathcal{A} \in \operatorname{Br}(X)$ such that $\mathcal{A} \mapsto A$ under the above map* $\operatorname{Br}(X) \to \operatorname{Br}(K)$.

Proof. **Step 1.** It is enough to show that there exists an open cover $X = \bigcup U_i$ and $\mathcal{A}_i \in \operatorname{Br}(U_i)$ for all i such that $\mathcal{A}_i \mapsto A$. Indeed, by Theorem 4.2, we have

$$0 \to \operatorname{Br}(U_i) \to \operatorname{Br}(K) \to \oplus_{x \in U_i^{(1)} \subset X^{(1)}} \mathrm{H}^1_{\mathrm{et}}(k(x), \mathbb{Q}/\mathbb{Z})$$

Let $\mathrm{res} : \mathrm{Br}(K) \to \oplus_{x \in X^{(1)}} \mathrm{H}^1_{\mathrm{et}}(k(x), \mathbb{Q}/\mathbb{Z})$ denote the map from the above diagram, so $\mathrm{res} = \prod_{x \in X^{(1)}} \mathrm{res}_x$ where $\mathrm{res}_x : \mathrm{Br}(K) \to \mathrm{H}^1_{\mathrm{et}}(k(x), \mathbb{Q}/\mathbb{Z})$. If there exists $\mathcal{A}_i \in \mathrm{Br}(U_i)$ that maps to A, then $\mathrm{res}_x(A) = 0$ for all $x \in U_i^{(1)}$. Thus if there exists $\mathcal{A}_i \in \mathrm{Br}(U_i)$ that maps to A for all i then $\mathrm{res}_x(A) = 0$ for all $x \in \bigcup U_i^{(1)} = X^{(1)}$. Thus $\mathrm{res}(A) = 0$, so there exists $\mathcal{A} \in \mathrm{Br}(X)$ such that $\mathcal{A} \mapsto A$.

Step 2. If $3 - x^2 \in \mathcal{O}^*_{U_i}$ where $\mathcal{O}^*_{U_i}$ denotes the units in $\mathcal{O}_{U_i}$, then there exists $\mathcal{A}_i \in \mathrm{Br}(U_i)$ such that $\mathcal{A}_i \mapsto A$. Indeed, let

$$\mathcal{A}_i := \mathcal{O}_{U_i}(1) \oplus \mathcal{O}_{U_i}(i) \oplus \mathcal{O}_{U_i}(j) \oplus \mathcal{O}_{U_i}(ij)$$

as an $\mathcal{O}_{U_i}$-module with algebra structure given by

$$\begin{aligned} \mathcal{A}_i \times \mathcal{A}_i &\to \mathcal{A}_i \\ i \times i &\mapsto 3 - x^2 \\ i \times j &\mapsto ij \\ j \times i &\mapsto -ji \\ j \times j &\mapsto -1 \\ &\cdots \end{aligned}$$

Then $\mathcal{A}_i$ is a quaternion algebra and hence is central simple over $\mathcal{O}_{U_i}$ and $A = \mathcal{A}_i \otimes_{\mathcal{O}_{U_i}} K$.

Step 3. Let $P_{(3-x^2)}$ be the point of $\mathbb{A}^1_{\mathbb{Q}} \subset \mathbb{P}^1_{\mathbb{Q}}$ corresponding to the prime ideal $(3 - x^2)$ of $\mathbb{Q}[x] = \mathcal{O}_{\mathbb{A}^1_{\mathbb{Q}}}$ and let $\infty := \mathbb{P}^1_{\mathbb{Q}} \setminus \mathbb{A}^1_{\mathbb{Q}}$. Define

$$U_A := X - \text{ the fiber above } \infty - \text{ the fiber above } P_{(3-x^2)}$$

Then $(3 - x^2) \in \mathcal{O}^*_{U_A}$ so there exists $\mathcal{A} \in \mathrm{Br}(U_A)$ such that $\mathcal{A} \mapsto A$. Indeed, we have

$$\begin{tikzcd} U_A \arrow[r, hook] \arrow[d] & X \arrow[d] \\ U \arrow[r, hook] & \mathbb{P}^1_{\mathbb{Q}} \end{tikzcd}$$

where $U = \mathbb{P}^1_{\mathbb{Q}} - \infty - P_{(3-x^2)}$, so

$$\begin{tikzcd} \Gamma(U_A, \mathcal{O}^*_{U_A}) \arrow[r, hook] & K(X) \\ \Gamma(U, \mathcal{O}^*_U) \arrow[r, hook] \arrow[u] & K(\mathbb{P}^1_{\mathbb{Q}}) \arrow[u] \end{tikzcd}$$

Since $3 - x^2$ is defined on $\mathbb{P}^1_{\mathbb{Q}} - \infty = \mathbb{A}^1_{\mathbb{Q}}$ and non-zero on $\mathbb{P}^1_{\mathbb{Q}} - \infty - P_{(3-x^2)} = \mathbb{A}^1_{\mathbb{Q}} - P_{(3-x^2)}$, we have $3 - x^2 \in \Gamma(U, \mathcal{O}^*_U)$. Since the map $\mathbb{Q}(x) = K(\mathbb{P}^1_{\mathbb{Q}}) \to K(X)$ simply takes x to x, $3 - x^2$ is a well-defined element of $\Gamma(U_A, \mathcal{O}^*_{U_A})$.

Step 4. Let B and C be the quaternion algebras over K given by $B = H(x^2 - 2, -1)$ and $C = H(\frac{3}{x^2} - 1, -1)$. Then $[A] = [B] = [C]$ as elements of $\mathrm{Br}(K)$.

Furthermore, let

$$U_B := X - \text{ the fiber above } \infty - \text{ the fiber above } P_{(x^2-2)}$$

and

$$U_C := X - \text{ the fiber above } 0 - \text{ the fiber above } P_{(3-x^2)}$$

Then we can apply precisely the same argument as in step 3 to show that there exist $\mathscr{B} \in \mathrm{Br}(U_B)$ and $\mathscr{C} \in \mathrm{Br}(U_C)$ such that $\mathscr{B} \mapsto B$ and $\mathscr{C} \mapsto C$.

Step 5. Notice that $X = U_A \bigcup U_B \bigcup U_C$. Thus A lifts to $\mathrm{Br}(X)$. □

Now we proceed to show that $X(\mathbb{A}_{\mathbb{Q}})^A = \emptyset$. Equivalently, we must show that for all $(x_\nu) \in X(\mathbb{A}_{\mathbb{Q}})$, we have $\sum_\nu \mathrm{inv}(\hat{x}_\nu(A)) \neq 0$. First, however, we will need some facts about cyclic algebras.

Let k be a field, let L/k be a finite, cyclic, Galois extension of degree n and let σ be the generator of $\mathrm{Gal}(L/k)$. We define an algebra A to be isomorphic to $L[x]$ as an additive group and with multiplication given by

$$x\ell = (\ell^\sigma)x \text{ for all } \ell \in L$$

Let $a \in k^\times$. Then $A/(x^n - a)$ is called a cyclic algebra and is denoted by (a, L).

Example 5. Let k be a global field, $L := k(\sqrt{-1})$ and $a \in k^\times$. Then $n = 2$ and $\mathrm{Gal}(L/k) \simeq \mathbb{Z}/2\mathbb{Z}$ generated by complex conjugation, denoted σ. The construction of A as above is $A := k(i)[j]$ with multiplication given by

$$\begin{aligned} j\ell &= \ell j \text{ for } \ell \in k, \text{ and} \\ ji &= (i^\sigma)j = -ij \end{aligned}$$

The corresponding cyclic algebra $A/(j^2 - a)$ is therefore isomorphic to the quaternion algebra over k given by

$$\begin{aligned} i^2 &= -1 \\ j^2 &= a \\ ij &= -ji \end{aligned}$$

Proposition 4.4. [Poo08, Proposition 1.4.21] *Let (a, L) be a cyclic algebra. Then $(a, L) \simeq M_n(k)$ if and only if $a \in \mathcal{N}_{L/k}(L^\times)$.*

Now let $(x_p) \in X(\mathbb{A}_{\mathbb{Q}})$, and let $\mathcal{A} \in \mathrm{Br}(X)$ be the lift of A that we constructed in Claim 4.3, considered as an Azumaya algebra on X. For each prime p, we have a map $x_p : \mathrm{Spec}\,\mathbb{Q}_p \to X$. So $\hat{x}_p(\mathcal{A})$ is simply the fiber of the sheaf $\mathcal{A}$ along the morphism x_p, i.e. $\hat{x}_p(\mathcal{A}) = \mathcal{A} \otimes_{\mathcal{O}_X} \mathbb{Q}_p$, which is by definition a central simple algebra over $\mathbb{Q}_p$. Indeed, if $x_p(\mathrm{Spec}\,\mathbb{Q}_p) \in U_A$ then $\hat{x}_p(\mathcal{A})$ is precisely the quaternion algebra given by

$$\begin{aligned} i^2 &= 3 - \underline{x}^2 \\ j^2 &= -1 \end{aligned}$$

where $\underline{x} \in \mathbb{Q}_p \cup \infty$ is the "x coordinate" of x_p i.e. the image of the rational function $x \in K(X)$ at the stalk of $x_p(\mathrm{Spec}\,\mathbb{Q}_p)$. Similarly, if $x_p(\mathrm{Spec}\,\mathbb{Q}_p) \in U_B$ then $\hat{x}_p(\mathcal{A})$

is the quaternion algebra given by

$$i^2 = \underline{x}^2 - 2$$
$$j^2 = -1$$

and if $x_p(\operatorname{Spec}\mathbb{Q}_p) \in U_C$ then $\hat{x}_p(\mathcal{A})$ is the quaternion algebra given by

$$i^2 = \frac{3}{\underline{x}^2} - 1$$
$$j^2 = -1$$

Note that $\hat{x}_p(\mathcal{A})$ is a cyclic algebra given by $a = 3 - \underline{x}^2$, $a = \underline{x}^2 - 2$, or $a = \frac{3}{\underline{x}^2} - 1$. Also, for any $Y \in \operatorname{Br}(\mathbb{Q}_p)$, we have $\operatorname{inv}(Y) = 0$ if and only if Y is trivial in $\operatorname{Br}(\mathbb{Q}_p)$, so $\operatorname{inv}(\hat{x}_p(\mathcal{A})) = 0$ if and only if $a \in \mathcal{N}_{\mathbb{Q}_p(i)/\mathbb{Q}_p}(\mathbb{Q}_p(i)^\times)$.

We split the computation of $\operatorname{inv}(\hat{x}_p(\mathcal{A}))$ into three cases:

Case 1: $p \neq 2, \infty$ If $v_p(\underline{x}) < 0$, then $\frac{3}{\underline{x}^2} \in \mathbb{Z}_p^\times$. If $v_p(\underline{x}) \geq 0$ then $3 - \underline{x}^2 \in \mathbb{Z}_p^\times$ or $\underline{x}^2 - 2 \in \mathbb{Z}_p^\times$. Indeed, if $v_p(3 - \underline{x}^2) > 0$ and $v_p(\underline{x}^2 - 2) > 0$ then $v_p(3 - \underline{x}^2 + \underline{x}^2 - 2) = v_p(1) > 0$, which is not true. Thus the structure constants for $\hat{x}_p(\mathcal{A})$ as a quaternion algebra over $\mathbb{Q}_p$ lie in $\mathbb{Z}_p^\times$. As in the proof of 3.2, we can construct an Azumaya algebra $A_p := \mathbb{Z}_p \oplus \mathbb{Z}_p i \oplus \mathbb{Z}_p j \oplus \mathbb{Z} - Pij$ with multiplication defined as for $\mathcal{A}$. Then $A_p \in \operatorname{Br}(\mathbb{Z}_p)$ since $A_p \otimes_{\mathbb{Z}_p} k(x)$ is a central simple algebra for all $x \in \operatorname{Spec}\mathbb{Z}_p$ since $p \neq 2$, so A_p is an Azumaya algebra on $\operatorname{Spec}\mathbb{Z}$. Furthermore $A_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathcal{A}$ so A_p maps to $\mathcal{A}$ under $\operatorname{Br}(\mathbb{Z}_p) \to \operatorname{Br}(\mathbb{Q}_p)$. But $\operatorname{Br}(\mathbb{Z}_p)$ is trivial, so $\mathcal{A}$ is trivial as an element of $\operatorname{Br}(\mathbb{Q}_p)$. So $\operatorname{inv}(\hat{x}_p(\mathcal{A})) = 0$.

Case 2: $p = \infty$

Here $x_p : \operatorname{Spec}\mathbb{R} \to X$. Equivalently, x_p is given by $(x, y, z) \in \mathbb{R}^3$ satisfying $(3 - x^2)(x^2 - 2) = y^2 + z^2$. In particular, $(3 - x^2)(x^2 - 2) \geq 0$, so either

$$(3 - x^2) \geq 0, \text{ and}$$
$$(x^2 - 2) \geq 0$$

or

$$(3 - x^2) \leq 0, \text{ and}$$
$$(x^2 - 2) \leq 0$$

This is only possible if $2 \leq x^2 \leq 3$. But then $(3 - \underline{x}^2) \geq 0$, and$(\underline{x}^2 - 2) \geq 0$ and at least one is nonzero, so at least one is in $\mathbb{R}_{>0} = \mathcal{N}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times)$. Thus $\hat{x}_p(\mathcal{A})$ is trivial and $\operatorname{inv}(\hat{x}_p(\mathcal{A})) = 0$.

Case 3: $p = 2$

If $v_2(\underline{x}) > 0$ then $3 - \underline{x}^2 \equiv 3 \equiv -1 \mod 4$

If $v_2(\underline{x}) = 0$ then $\underline{x}^2 \equiv 1 \mod 4$ so $\underline{x}^2 - 2 \equiv -1 \mod 4$

If $v_2(\underline{x}) < 0$ then $\frac{3}{\underline{x}^2} - 1 \equiv -1 \mod 4$

But if $z \in \mathcal{N}_{\mathbb{Q}_2(i)/\mathbb{Q}_2}(\mathbb{Q}_2(i))$ then $z = (a+bi)(a-bi) = a^2 + b^2$ for some 2-adic integers a, b so $z \equiv 0, 1 \mod 4$. Thus regardless of the value of $v_2(\underline{x})$, $\mathrm{inv}(\hat{x}_2(\mathcal{A})) \neq 0$.

Thus we have $\sum_p \mathrm{inv}(\hat{x}_p(\mathcal{A})) = \mathrm{inv}(\hat{x}_2(\mathcal{A})) \neq 0$ for all adelic points $(x_p) \in X(\mathbb{A}_{\mathbb{Q}})$. So $X(\mathbb{A}_{\mathbb{Q}})^A = \emptyset$ so $X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} = \emptyset$. Thus X is an example of a scheme for which $X(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}} = \emptyset$ and hence $X(k) = \emptyset$, but $X(\mathbb{A}_k) \neq \emptyset$.

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