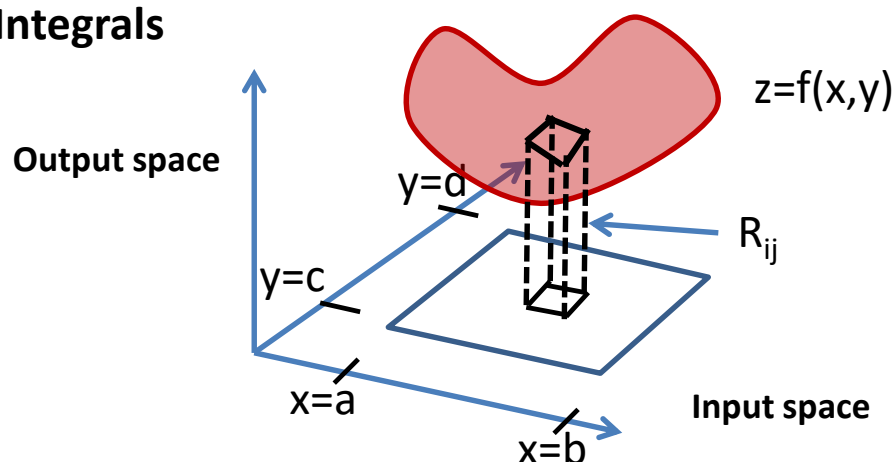


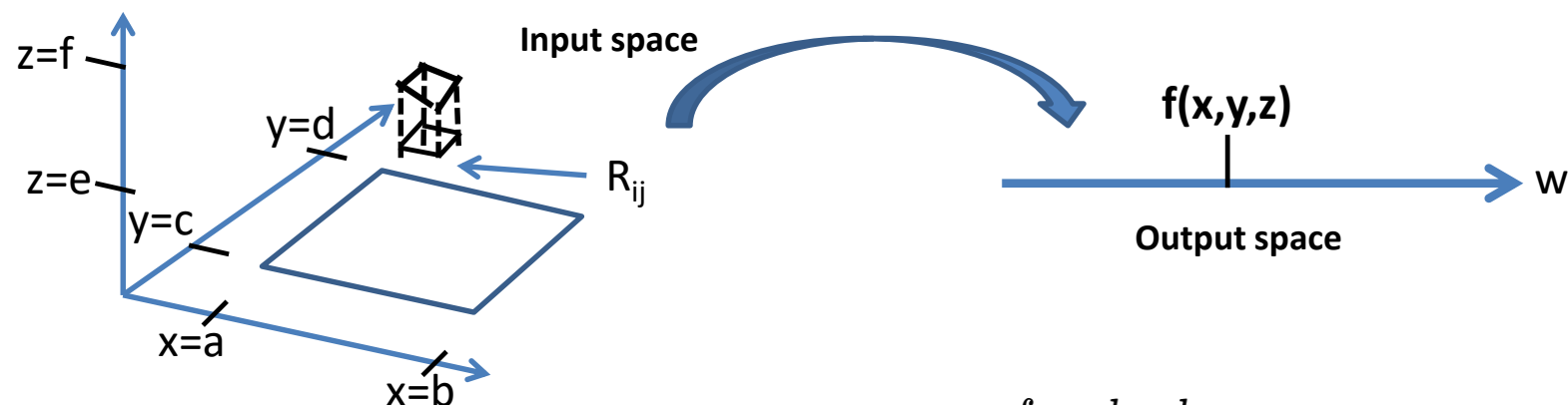
Section 15.7: Triple Integrals

So far, we have double integrals:

$$\text{Volume} = \int_c^d \int_a^b f(x, y) dx dy$$



We can do the same thing for triple integrals of $f(x,y,z)$



$$\text{HyperVolume} = \int_e^f \int_c^d \int_a^b f(x, y, z) dx dy dz$$

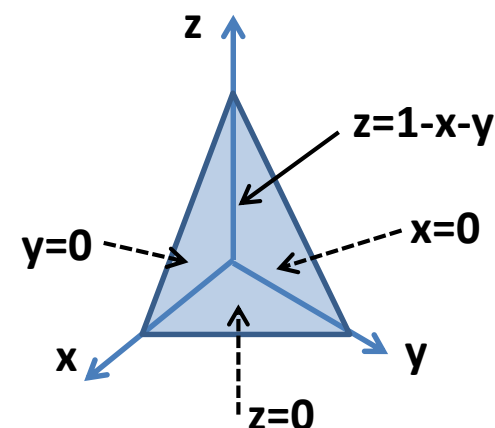
Section 15.7: Triple Integrals

Okay—what about triple integrals over weird regions?

Consider the tetrahedron built from four planes:

Plane $x=0$, plane $y=0$, plane $z=0$, plane $x+y+z=1$

What is the integral of $f(x,y,z)=z$ over this volume?



We follow the same idea:

We pick an outer region of integration (Reg:1), followed by an inner region (Reg:2) and then an even more inner region (Reg:3)

$$\int f(x, y, z) dx dy dz = \int_{Reg:1low}^{Reg:1up} \int_{Reg:2low}^{Reg:2up} \int_{Reg:3low}^{Reg:3up} f(x, y, z) d(Reg : 3) d(Reg : 2) d(Reg : 1)$$

Section 15.7: Triple Integrals

And now we do the integral:

It's clear that x goes from $x=0$ to $x=1$

$$\int f(x, y, z) dx dy dz = \int_0^1 \left[\int_{Reg:2low}^{Reg:2up} \int_{Reg:3low}^{Reg:3up} f(x, y, z) dz dy \right] dx$$

Step 2: For any value of x , what are the limits on y ?

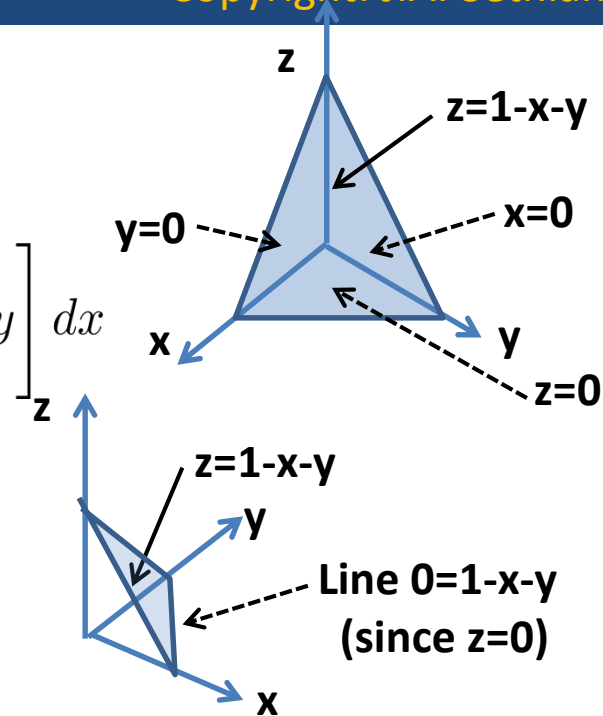
For any x , y starts at $y=0$ and ends at $y=1-x$

$$\int f(x, y, z) dx dy dz = \int_0^1 \left[\int_0^{1-x} \left[\int_{Reg:3low}^{Reg:3up} f(x, y, z) dz \right] dy \right] dx$$

Step 3: For any value of x and y , what are the limits on z ?

For any x and y , z starts at $z=0$ and ends at $z=1-x-y$

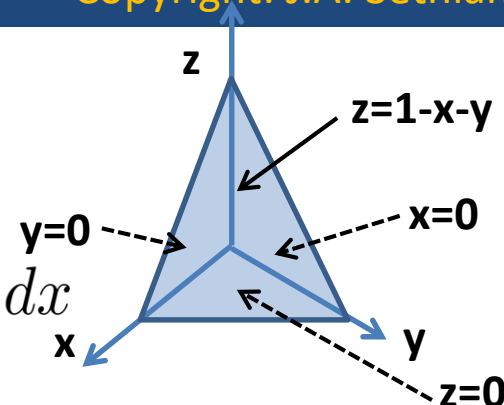
$$\int f(x, y, z) dx dy dz = \int_0^1 \left[\int_0^{1-x} \left[\int_0^{1-x-y} f(x, y, z) dz \right] dy \right] dx$$



Section 15.7: Triple Integrals

Step 1: Let's pick x as the outer region

$$\int f(x, y, z) dx dy dz = \int_0^1 \left[\int_0^{1-x} \left[\int_0^{1-x-y} z dz \right] dy \right] dx$$



$$= \int_0^1 \left[\int_0^{1-x} \left[\left. \frac{z^2}{2} \right|_0^{1-x-y} \right] dy \right] dx = \int_0^1 \left[\int_0^{1-x} \left[\frac{(1-x-y)^2}{2} \right] dy \right] dx$$

$$= \int_0^1 \left[\left. \frac{-(1-x-y)^3}{6} \right|_0^{1-x} \right] dx = \int_0^1 \frac{(1-x)^3}{6} dx = \frac{1}{6} \left[\left. \frac{(1-x)^4}{4} \right|_0^1 \right] = \frac{1}{24}$$

Section 15.7: Triple Integrals

We still have Fubini's Theorem: If $f(x,y,z)$ is continuous, then

$$\begin{aligned} \text{Volume} &= \int_e^f \int_c^d \int_a^b f(x,y,z) dx dy dz = \int_e^f \left[\int_c^d \left[\int_a^b f(x,y,z) dx \right] dy \right] dz \\ &= \int_e^f \left[\int_a^b \left[\int_c^d f(x,y,z) dy \right] dx \right] dz = \int_a^b \left[\int_c^d \left[\int_e^f f(x,y,z) dz \right] dy \right] dx \end{aligned}$$

..... You get the idea...

Example: Find the integral of $f(x,y,z) = x y z^2$ over the box $0 \leq x \leq 1; -1 \leq y \leq 2; 0 \leq z \leq 3$;

Step 1: Set it up

$$\begin{aligned} \int_e^f \int_c^d \int_a^b f(x,y,z) dx dy dz &= \int_0^3 \int_{-1}^2 \int_0^1 x y z^2 dx dy dz \\ &= \int_0^3 \int_{-1}^2 \left[\left| \frac{x^2 y z^2}{2} \right|_0^1 \right] dy dz = \int_0^3 \int_{-1}^2 \left[\frac{y z^2}{2} \right] dy dz = \int_0^3 \left[\left| \frac{y^2 z^2}{4} \right|_{-1}^2 \right] dz \\ &= \int_0^3 \left[\frac{4z^2}{4} - \frac{z^2}{4} \right] dz = \int_0^3 \left[\frac{3z^2}{4} \right] dz = \left| \frac{3z^3}{4*3} \right|_0^3 = \frac{27}{4} \end{aligned}$$

Section 15.7: Triple Integrals

Obviously, the hard part of all this is setting up the integrals. Make pictures!!!!

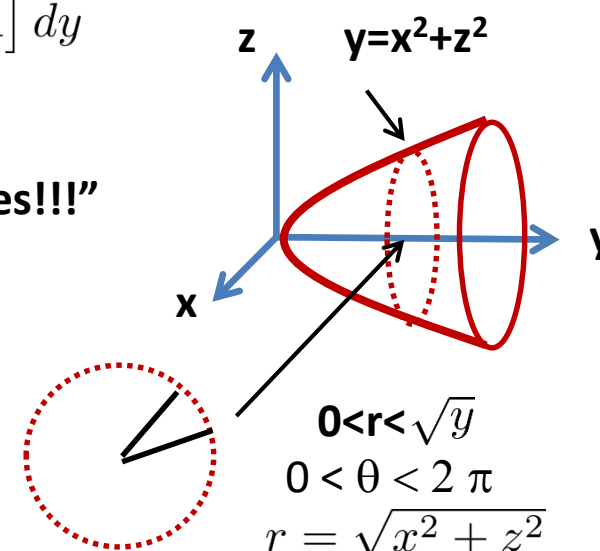
Example: Evaluate the integral of $\sqrt{x^2 + z^2}$
over the region E bounded by the paraboloid
 $y = x^2 + z^2$ and the plane $y = 4$ (????????????????)

Idea #1: Let y go from $y = 0$ to $y = 4$ $\int_0^4 \left[\int_A \sqrt{x^2 + z^2} dA \right] dy$

Idea #2: That inner integral is screaming “polar coordinates!!!”

At value y , the slice has radius $\sqrt{y}$

So, we have $x = r \cos \theta$, $z = r \sin \theta$ so $dx dz = r dr d\theta$
and the radius r goes from 0 to $\sqrt{y}$



$$\begin{aligned} \int_0^4 \left[\int_A \sqrt{x^2 + z^2} dA \right] dy &= \int_0^4 \left[\int_0^{2\pi} \left[\int_0^{\sqrt{y}} r r dr \right] d\theta \right] dy \\ &= \int_0^4 \left[\int_0^{2\pi} \left[\left|_0^{\sqrt{y}} \frac{r^3}{3} \right] d\theta \right] dy = \int_0^4 \left[\int_0^{2\pi} \frac{y^{3/2}}{3} d\theta \right] dy = \int_0^4 \left[2\pi \frac{y^{3/2}}{3} \right] dy = \frac{2\pi}{3} \left|_0^4 \frac{2}{5} y^{5/2} \right. \\ &= \frac{2\pi}{3} \frac{2}{5} 4^{5/2} \end{aligned}$$

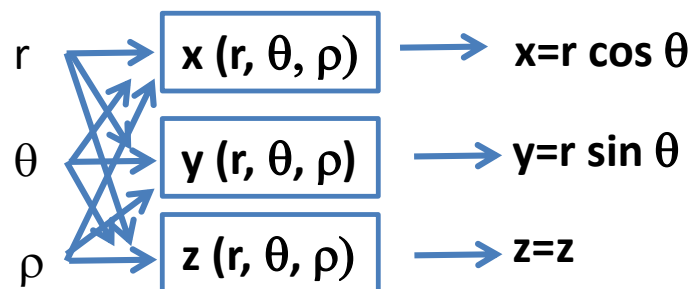
Section 15.8: Triple Integrals: Cylindrical and Spherical Coordinates

What about other 3d coordinate systems besides Cartesian?

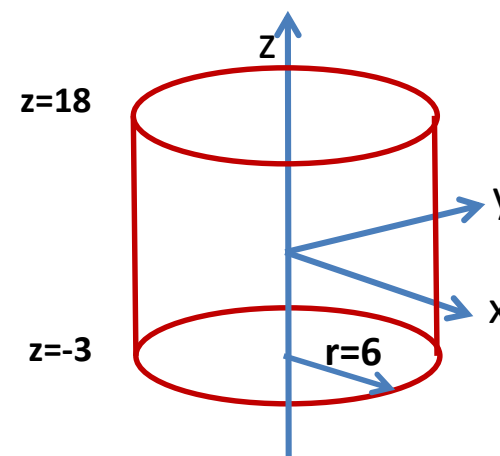
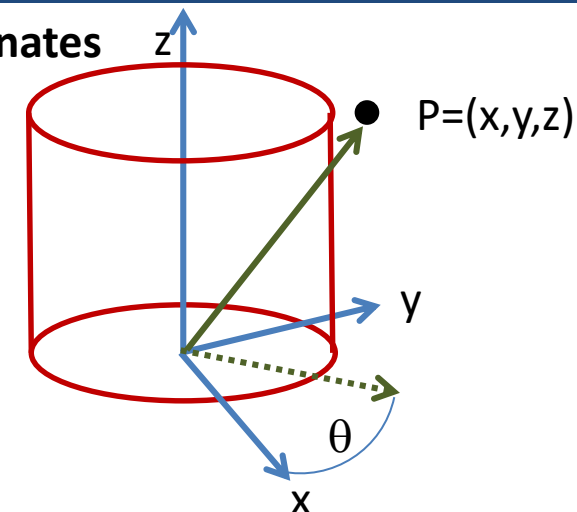
(1) Cylindrical coordinates (r, θ, z)

r and θ are polar coordinates of the projection of the point $P=(x,y,z)$ on the x - y plane. And z is the height

So, $x=r \cos \theta$, $y=r \sin \theta$, and $z=z \Rightarrow r = \sqrt{x^2 + y^2}$, $z = z$



Example: $r=6$, $-3 < z < 18$ is the set of all point lying on a cylinder of radius 6 with bottom at $z=-3$ and top at $z=18$



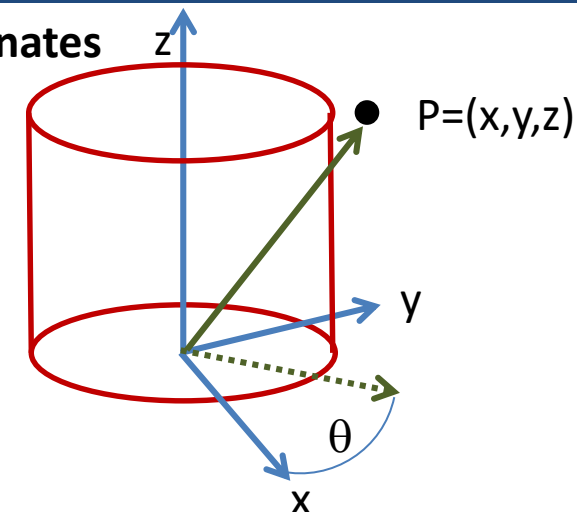
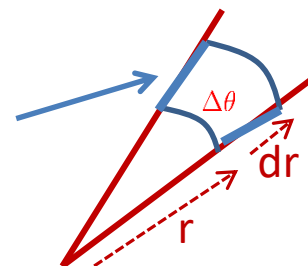
Section 15.8: Triple Integrals: Cylindrical and Spherical Coordinates

Cylindrical coordinates (r, θ, z)

Okay, what is the differential element?

Recall:

$$dx dy = r dr d\theta$$



So, just extending z upwards, we must have that the differential element is

$$dx dy dz = r dr d\theta dz$$

Cartesian
element

Cylindrical
element

Example: Find the volume of a cylinder of radius 4 and height 6

(you know the answer: Volume = cross-section $\times$ height = $\pi r^2 H = \pi (16)(6) = 96 \pi$)

$$\begin{aligned} \text{Volume} &= \int_{\text{cylinder}} f(x, y, z) dx dy dz = \int_0^6 \int_0^{2\pi} \int_0^4 f(x, y, z) r dr d\theta dz = \int_0^6 \int_0^{2\pi} \int_0^4 1 r dr d\theta dz \\ &= \int_0^6 \int_0^{2\pi} 8 d\theta dz = 6 * 2\pi 8 = 96\pi \end{aligned}$$

Section 15.9: Triple Integrals: Spherical Coordinates

Spherical coordinates (ρ, θ, ϕ) $0 \leq \rho$

$$0 \leq \theta \leq 2\pi$$

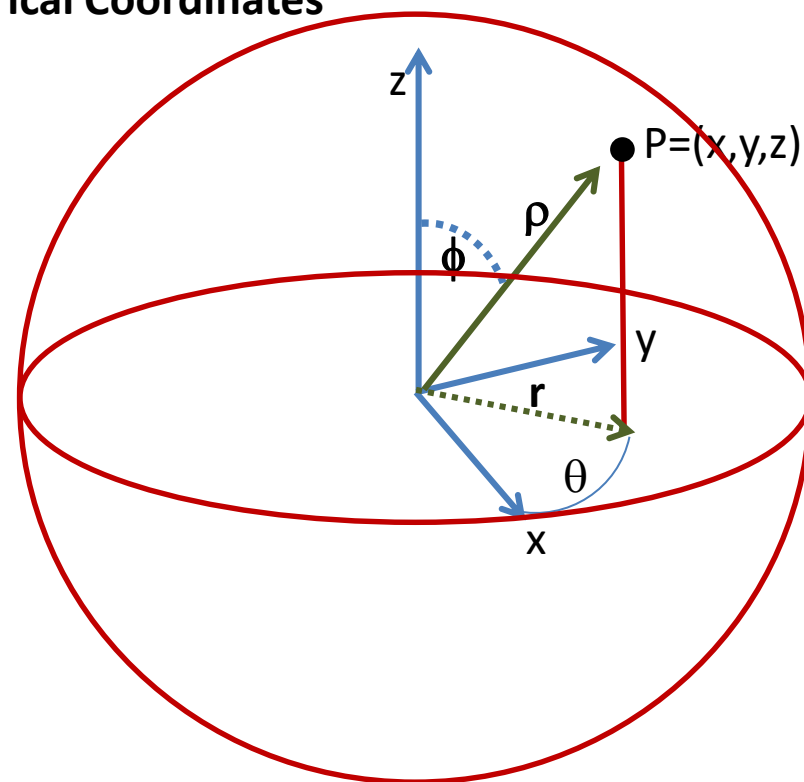
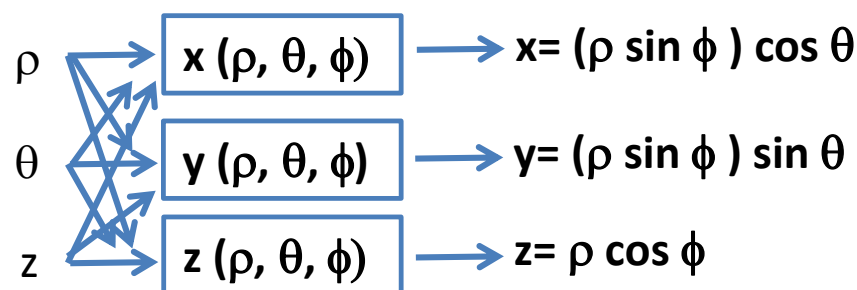
$$0 \leq \phi \leq \pi$$

Let's find the conversion

$$x = r \cos \theta = (\rho \sin \phi) \cos \theta$$

$$y = r \sin \theta = (\rho \sin \phi) \sin \theta$$

$$z = \rho \cos \phi$$



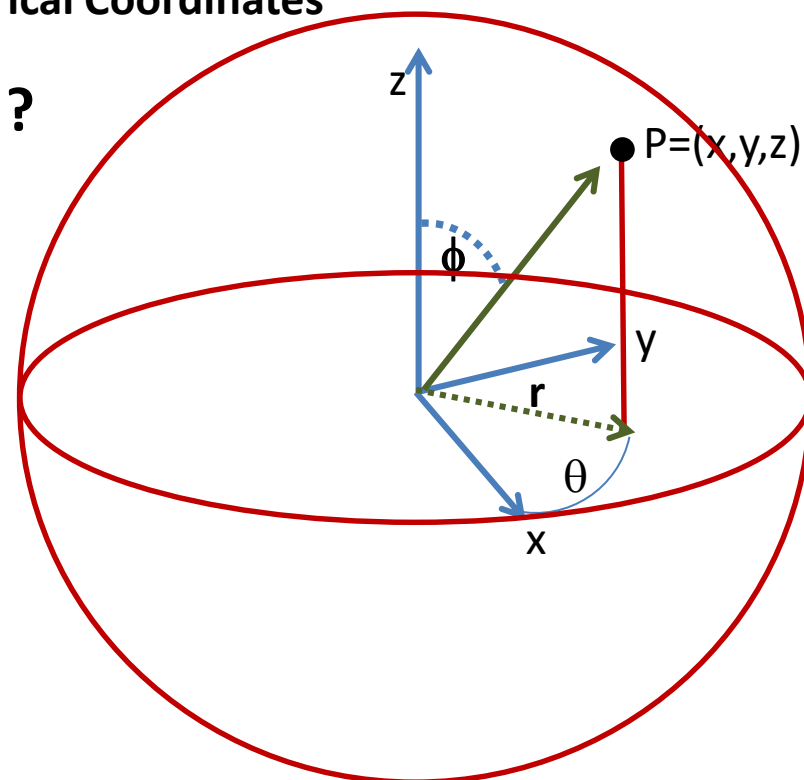
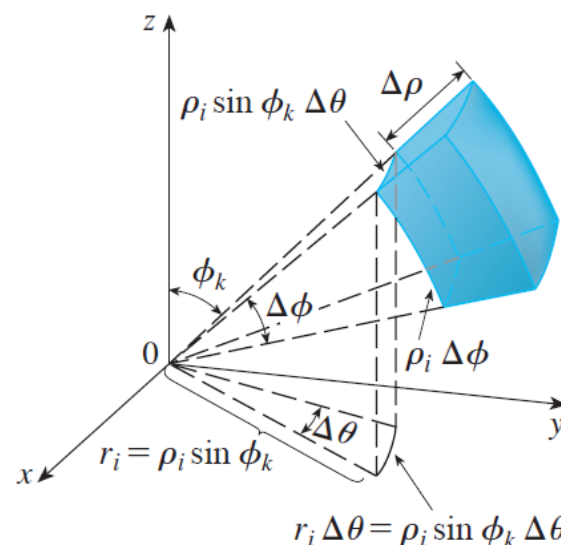
Example: What is the object $\rho=9$?

Answer: a sphere of radius 9

Section 15.9: Triple Integrals: Spherical Coordinates

So what is the differential element $dx\,dy\,dz$?

Taken from
Stewart



There is a long explanation in the book---I am going to tell you the answer:

$$dx\,dy\,dz = \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$$

Please read the explanation in the book—but I'm going to give you a much neater way in the next lecture!