

Section 15.3: To summarize

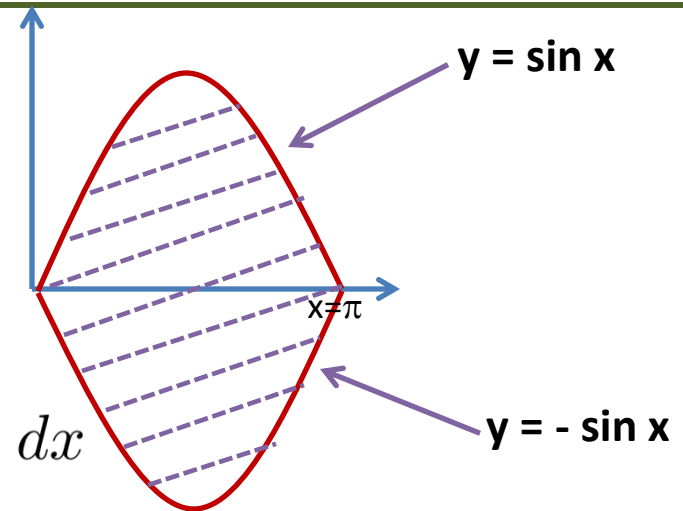
Integrate $f(x,y) = 3$ over the regions between $y=\sin x$, $y=-\sin x$, $x=0$ and $x=\pi$

Solution:

Step 1: Draw a **picture**

Step 2: Set up the integral

$$\int_{x \text{ start}}^{x \text{ end}} \left[\int_{y \text{ start}}^{y \text{ end}} dy \right] dx = \int_0^\pi \left[\int_{-\sin x}^{\sin x} 3 dy \right] dx$$



Step 3: Do the integral

$$\begin{aligned} \int_0^\pi \left[\int_{-\sin x}^{\sin x} 3 dy \right] dx &= \int_0^\pi \left[\left. 3y \right|_{-\sin x}^{\sin x} \right] dx \\ &= \int_0^\pi [6 \sin x] dx = \left| -6 \cos x \right|_0^\pi = 6 - (-6) = 12 \end{aligned}$$

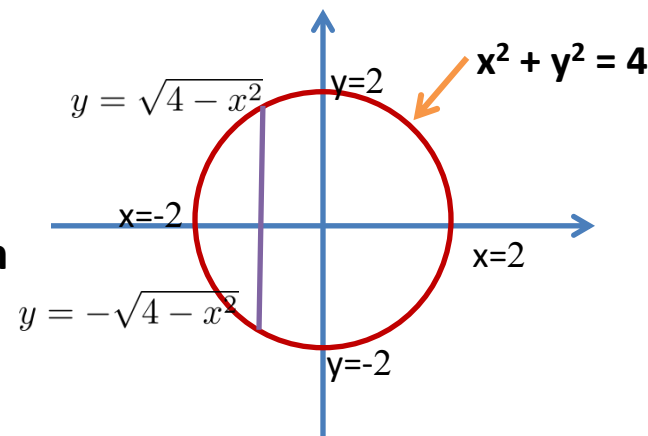
What if I asked for the area inside the circle $x^2 + y^2 = 4$?

Step 1: Well, you could draw it:

Step 2: And then, given x between -2 and 2 , y goes from

$$\text{y start} = y = -\sqrt{4 - x^2}$$

$$\text{y end} = y = \sqrt{4 - x^2}$$



Step 3: And then you could set up the integral:

$$\int_{x \text{ start}}^{x \text{ end}} \left[\int_{y \text{ start}}^{y \text{ end}} dy \right] dx = \int_{-2}^2 \left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 1 \, dy \right] dx$$

Step 4: And then you could do the integral:

$$= \int_{-2}^2 \left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 1 \, dy \right] dx = \int_{-2}^2 \left[\left. y \right|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \right] dx$$

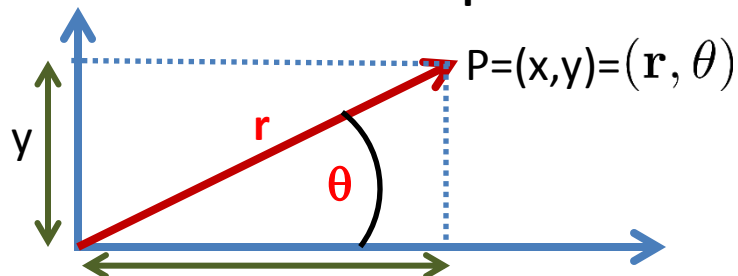
$$= \int_{-2}^2 [2\sqrt{4 - x^2}] dx$$

And I'm too lazy to do this integral. Any other ideas?

Section 15.4: Polar integration:

Area inside the circle $x^2 + y^2 = 4$?

A different idea: Let's use polar coordinates



So what is the area of each sliver?

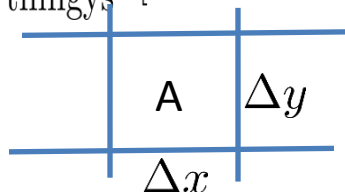
First, we divide the sliver into pieces

- (1) Let $\Delta\theta$ be the change in the angle as we sweep out the curved box
- (2) Let Δr be the change in the radius as we sweep out the curved box

What is the area of this thingy?

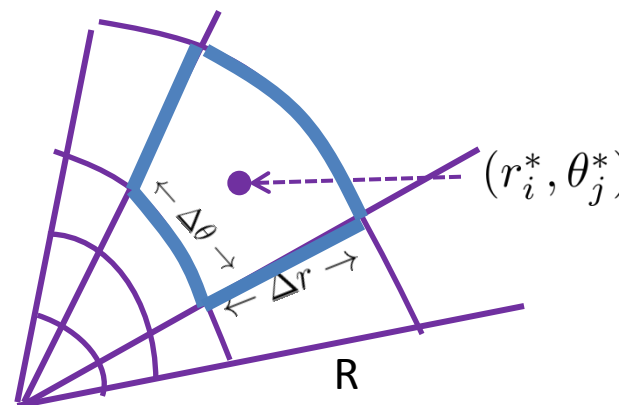
$$\text{Area} = \sum_{\text{all "thingys"}} [\text{area of each "thingy"}]$$

In normal Cartesian coordinates



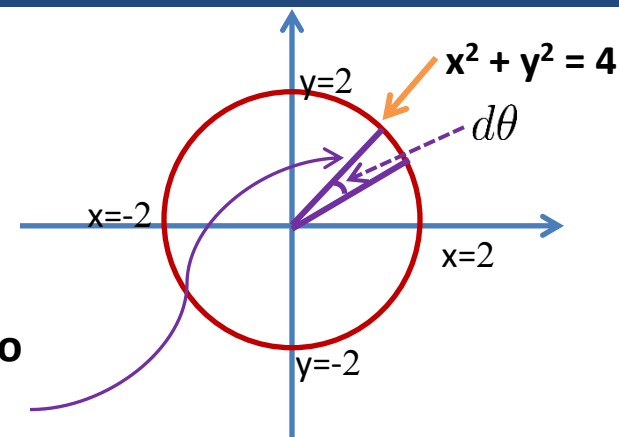
$$\text{Area} = \Delta x \Delta y$$

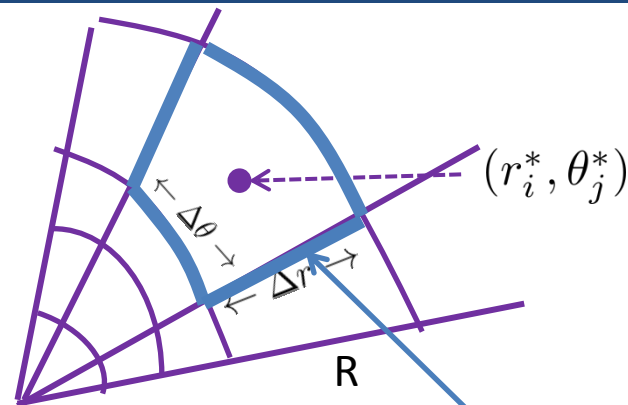
To compute the area, we want to add up "slivers"



In this new coordinate system, we need to figure out the area of the curved box

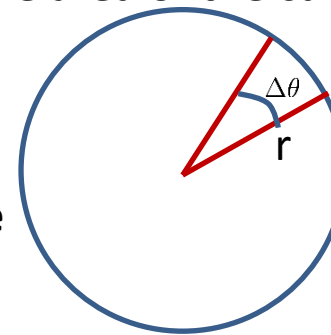
And it's **not** $\text{Area} = \Delta r \Delta\theta$!!!
(Now I need to explain why....)





Figuring out the area of the curved box....

Step 1: Let's remember the formula for the area of a sliver

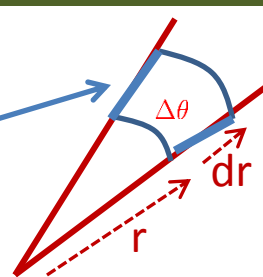


Area of whole circle = πr^2

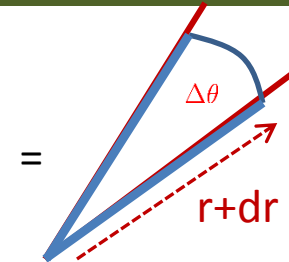
So area of sliver is

$$\left(\frac{\Delta\theta}{2\pi}\right) \pi r^2 = \frac{r^2}{2} \Delta\theta$$

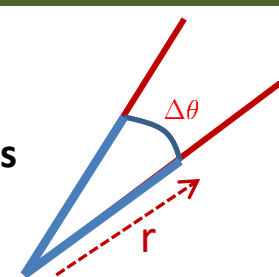
And now we can find the area of the curved box



Curved box



minus



= Whole thing - Shorter thing

$$= \frac{1}{2}(r+dr)^2 d\theta - \frac{1}{2}r^2 d\theta$$

$$= \frac{1}{2}(r^2 + 2rdr + (dr)^2) d\theta - \frac{1}{2}r^2 d\theta$$

$$= \boxed{rdrd\theta} + \boxed{\frac{1}{2}(dr)^2 d\theta}$$

Term in red box goes to zero faster than term is purple box

So the area of the curved box is $AREA = rdrd\theta$

So the area of the curved box is $AREA = r dr d\theta$

Let's use it: Find the area of a circle of radius 4 using polar integration (yes, the answer is $\pi (4)^2$)

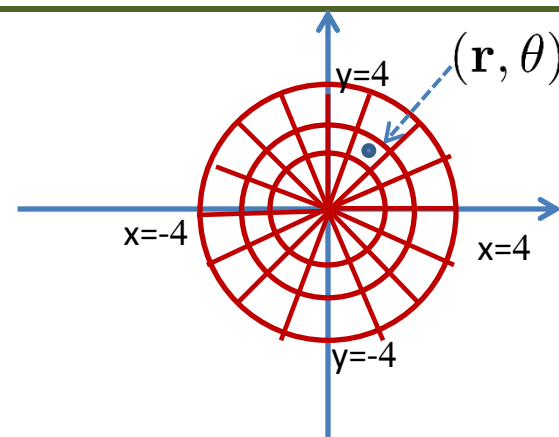
Step 1: We choose θ as the outer variable of integration, and r as the inner variable

$$\text{Area} = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=4} f(r, \theta) r dr d\theta$$

Step 2: The function we are integrating is just $f(r, \theta)=1$, since it is the area

Step 3: Do the integral

$$\text{Area} = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=4} 1 r dr d\theta = \int_{\theta=0}^{\theta=2\pi} \left[\frac{r^2}{2} \right]_{r=0}^{r=4} d\theta = \int_{\theta=0}^{\theta=2\pi} 8 d\theta = \left[8\theta \right]_{\theta=0}^{\theta=2\pi} = 16\pi$$



So the area of the curved box is $AREA = r dr d\theta$

Suppose we did it the other way

Step 1: We choose r as the outer variable of integration, and π as the inner variable

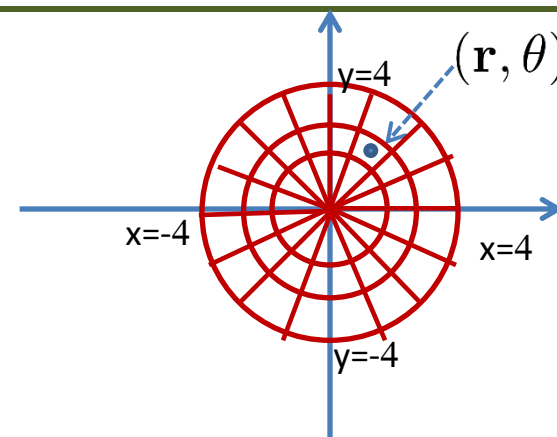
$$\text{Area} = \int_{r=0}^{r=4} \int_{\theta=0}^{\theta=2\pi} f(r, \theta) r d\theta dr$$

Step 2: The function we are integrating is just $f(r, \theta)=1$, since it is the area

Step 3: Do the integral

$$\text{Area} = \int_{r=0}^{r=4} \int_{\theta=0}^{\theta=2\pi} 1 r d\theta dr = \int_{r=0}^{r=4} r \left[\theta \right]_{\theta=0}^{\theta=2\pi} dr = \int_{r=0}^{r=4} 2\pi r dr = \left. 2\pi r^2 / 2 \right|_{r=0}^{r=4} = 16\pi$$

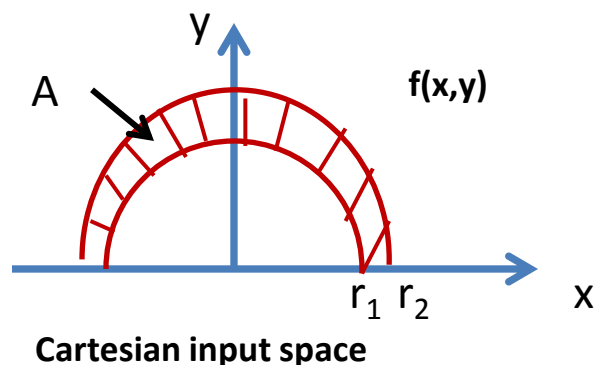
↑
(note that we can pull r
out of this integral)



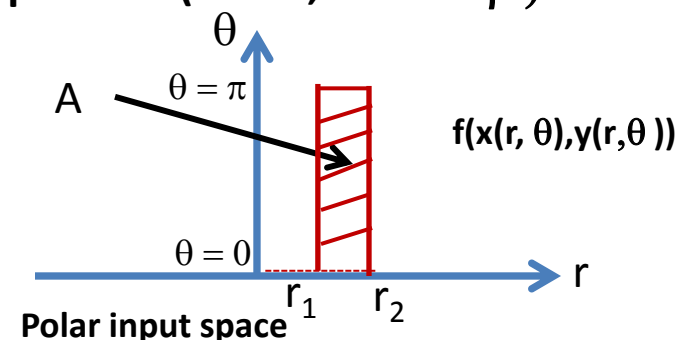
So the area of the curved box is $AREA = r dr d\theta$

So, let's repeat:

Suppose we have a region A in xy input space, and a function $f(x,y)$ defined in input space:



Suppose we could describe the same region in polar input space: $A = (a < r < b, \alpha < \theta < \beta)$



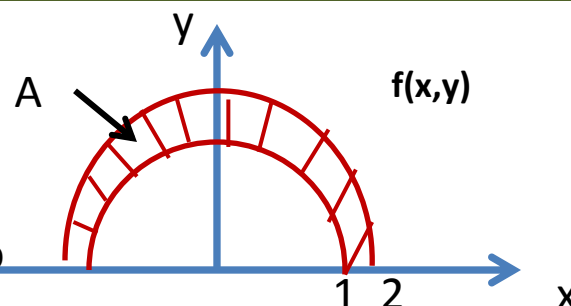
Then we can transform the integral from Cartesian coordinates to polar coordinates:

$$\int \int_A f(x, y) dx dy = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta$$

Let's use this ---

Example: Set up the integral to evaluate the function $f(x,y) = 3x + 4y^2$ over the region R where R is the region in the upper half plane bounded by the circles $x^2+y^2 = 1$ and $x^2+y^2=4$

Step 1: Draw the region—this is **always** the first step



So r goes from 1 to 2

Step 2: Decide which you want to be the outer integral

I choose θ , θ goes from 0 to π

Step 3: Decide which you want to be the inner integral

I choose r , r goes from 1 to 2

Step 4: Write down the integral

$$\iint_A (3x + 4y^2) dx dy = \int_0^\pi \left[\int_1^2 (3x + 4y^2) r dr \right] d\theta = \int_0^\pi \left[\int_1^2 (3(r \cos \theta) + 4(r \sin \theta)^2) r dr \right] d\theta$$

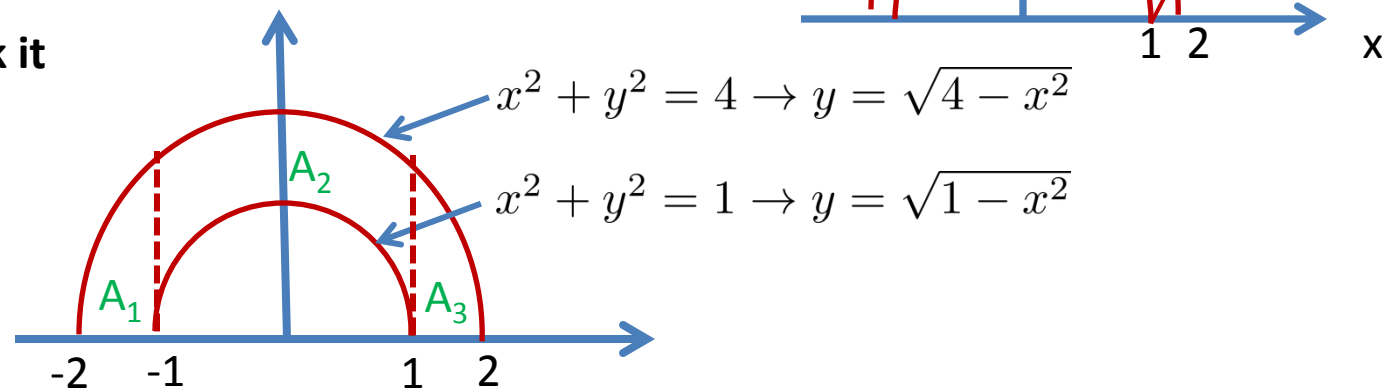
Don't forget to write x and y in terms of the polar coordinates

Step 5: Do the integral---I will leave this to you

$$\int \int_A (3x + 4y^2) dx dy = \int_0^\pi \left[\int_1^2 (3x + 4y^2) r dr \right] d\theta = \int_0^\pi \left[\int_1^2 (3(r \cos \theta) + 4(r \sin \theta)^2) r dr \right] d\theta$$

I admit—this integral doesn't like fun to integrate—but look how much worse it would be in Cartesian coordinates!

We would need to break it up into three regions



$$\text{Area} = \text{Area}_1 + \text{Area}_2 + \text{Area}_3$$

$$= \int_{-2}^{-1} \int_{y=0}^{y=\sqrt{4-x^2}} (2x + 4y^2) dy dx + \int_{-1}^1 \int_{y=\sqrt{1-x^2}}^{y=\sqrt{4-x^2}} (2x + 4y^2) dy dx + \int_1^2 \int_{y=0}^{y=\sqrt{4-x^2}} (2x + 4y^2) dy dx$$

And I really don't want to do this integral.....

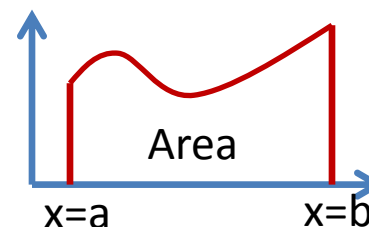
Some final comments —be careful about direction:

θ, r increasing gives $r dr d\theta$ just like x, y increasing gives $dx dy$

And let's make sure we understand one more thing

Question: Find the area under the curve $y=f(x)$ between $x=a$ and $x=b$

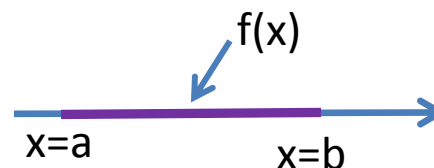
This one wants you to integrate the area under a function graphed with output against input



$$\text{Area} = \int_a^b f(x) dx$$

Question: Integrate the function $f(x)$ along the x axis between $x=a$ and $x=b$

This one wants you to integrate the total “weight” of a function describe in input space.



$$\text{Answer} = \int_a^b f(x) dx$$

Note that they are the same thing...

Before I go further, let me show you something **amazing**

Definition: A matrix is a table of numbers

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

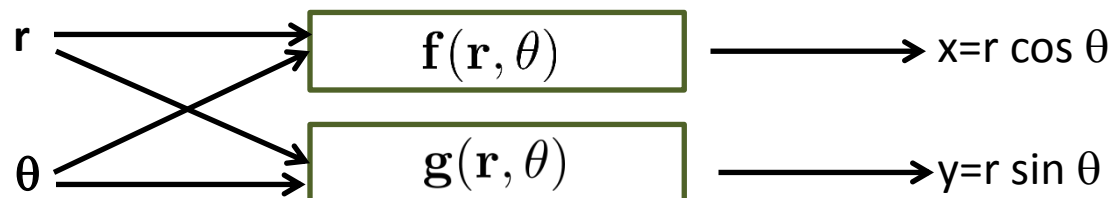
$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

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Definition: The determinant of a 2x2 matrix is $\det A = ad - bc$

Definition: The determinant of a 3x3 matrix is $\det B = a(ei - fh) - b(di - fg) + c(dh - eg)$

Polar coordinates



Suppose we form the “matrix of partials”

$$A = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$$

$$\det A = (\cos \theta)(r \cos \theta) - (\sin \theta)(-r \sin \theta) = r(\cos^2 \theta + \sin^2 \theta) = r$$

It seems that $dx dy = \det A dr d\theta = r dr d\theta$ WOW!!!!!! (don't worry—I will prove this later!)

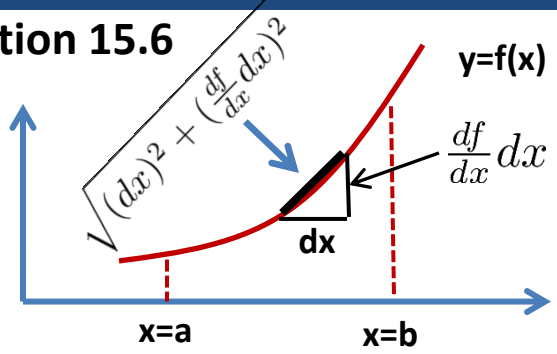
Section 15.5

Please read this section, and do the assigned homework problems....

Surface Area

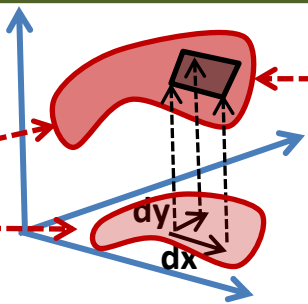
Section 15.6

In 1D: the length of the curve from $(a,f(a))$ to $(b,f(b))$ is



Length = $\int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2} \, dx$

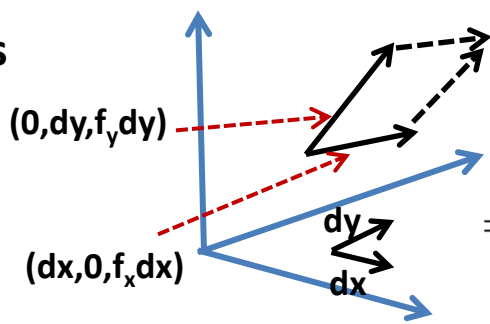
What is the area of a surface $z=f(x,y)$ that is the graph of an input region?



Plan: divide surface into a bunch of small surface patches

Area of patch = dSurfArea
How can we find the area of a surface patch?

Partial derivatives tell us how the sides of the patch change:



And we know the area of a parallelogram: it is

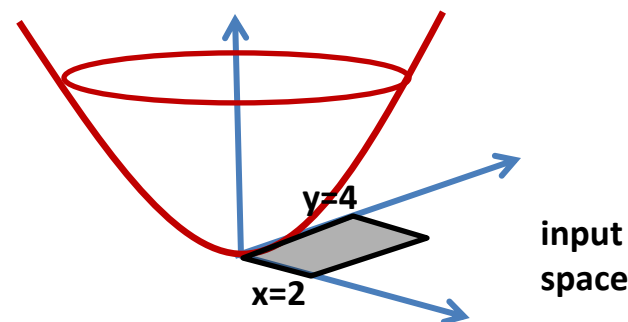
$$\begin{aligned} |\vec{u} \times \vec{v}| &= |(dx, 0, f_x dx) \times (0, dy, f_y dy)| \\ &= \begin{vmatrix} i & j & k \\ dx & 0 & f_x dx \\ 0 & dy & f_y dy \end{vmatrix} = |(f_x dx dy, -f_y dx dy, dx dy)| \\ &= \sqrt{(f_x dx dy)^2 + (-f_y dx dy)^2 + (dx dy)^2} \\ &= \sqrt{1 + (f_x)^2 + (f_y)^2} \, dx dy \end{aligned}$$

$\int_{\text{input region}} \sqrt{1 + (f_x)^2 + (f_y)^2} \, dx dy$

They are the same!

$$\int_{\text{input region}} \sqrt{1 + (f_x)^2 + (f_y)^2} \, dx dy$$

Surface Area Example: Write down the integral representing the surface area of $z=x^2+2y^2$ above the patch $0 \leq x \leq 2$, $0 \leq y \leq 4$



Let's use y as the outside variable

Then y goes from 0 to 4, and the inner variable x goes from 0 to 2

$$\int_0^4 \int_0^2 \sqrt{1 + (f_x)^2 + (f_y)^2} \, dx dy = \int_0^4 \int_0^2 \sqrt{1 + (2x)^2 + (4y)^2} \, dx dy$$