

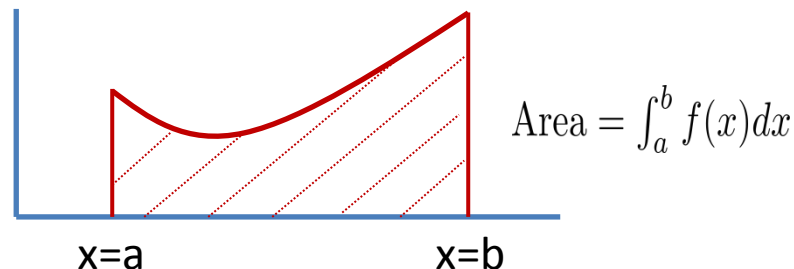
Section 15.1-15.2: On to integration!

- We've just done differentiation
- Obviously, what comes next is integration (Chapter 15)
- And after that, Chapter 16 is the

Fundamental theorems of multidimensional calculus

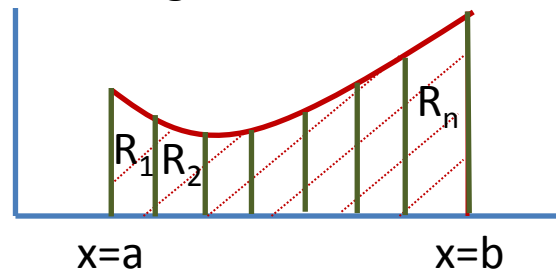
So let's begin: The good news: this looks a lot like everything you learned in 1D

Let's recall 1D $\int_a^b f(x)dx =$ Area under the curve $y=f(x)$ between $x=a$ and $x=b$



And what is that really?

Answer: the sum of rectangles



$$\text{Area} = \int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n R_i = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^n R_i$$

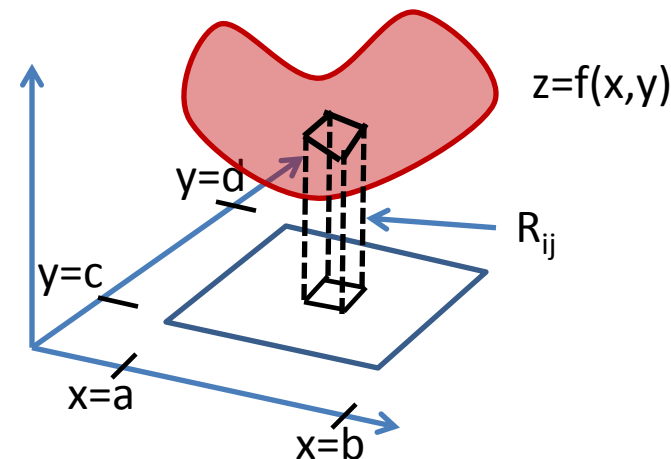
And you already knew that....

Section 15.1: On to integration!

Well, this should come as no surprise:

$$\text{Volume} = \int_c^d \int_a^b f(x,y) dx dy = \lim_{\Delta x \rightarrow 0} \lim_{\Delta y \rightarrow 0} \sum_{i=1, j=1}^n R_{ij}$$

Instead of summing up the area of rectangles, you sum up the volume of skyscrapers



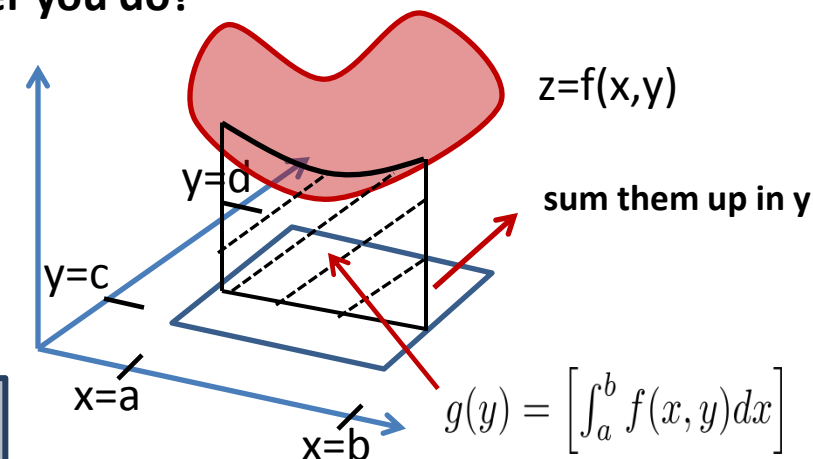
First interesting question: Does it matter what order you do?

Does $\int_c^d \int_a^b f(x,y) dx dy = \int_a^b \int_c^d f(x,y) dy dx$?

More precisely: $\int_c^d \int_a^b f(x,y) dx dy = \int_c^d \left[\int_a^b f(x,y) dx \right] dy$

This is a function of y: call it g(y)

$$= \int_c^d g(y) dy$$



Fubini's Theorem: if $f(x,y)$ is continuous, then

they are equal!!! $\int_c^d \int_a^b f(x,y) dx dy = \int_a^b \int_c^d f(x,y) dy dx$

Let's use all this. **Example: Find** $\int_0^3 \int_1^2 x^2 y dx dy$

**Step 1: Rewrite
with brackets**

$$\int_0^3 \int_1^2 x^2 y dx dy = \int_0^3 \left[\int_1^2 x^2 y dx \right] dy$$

**Step 2: Do the
inner integral,
holding y fixed**

$$\left[\int_1^2 x^2 y dx \right] = \left| \frac{1}{3} x^3 y \right|_1^2 = \frac{1}{3} (2^3) y - \frac{1}{3} (1^3) y = \frac{1}{3} (8 - 1) y = \frac{7}{3} y = g(y)$$

**Step 3: Put the result of the inner
integral back in and finish**

$$\int_0^3 \int_1^2 x^2 y dx dy = \int_0^3 g(y) dy = \int_0^3 \frac{7}{3} y dy = \left| \frac{7}{3 \cdot 2} y^2 \right|_0^3 = \frac{7}{6} (3^2 - 0^2) = \frac{63}{6}$$

Done

And because Fubini's theorem applies, let's check that it doesn't matter which way you do it

**Step 1: Swap
the order of
integration**

$$\int_1^2 \int_0^3 x^2 y \, dy dx = \int_1^2 \left[\int_0^3 x^2 y \, dy \right] dx$$

**Step 2: Do the
inner integral,
holding y fixed**

$$\left[\int_0^3 x^2 y \, dy \right] = \left| \frac{x^2 y^2}{2} \right|_0^3 = \frac{1}{2}(3^2)x^2 - \frac{1}{2}(0^2) = \frac{9}{2}x^2 = g(x)$$

**Step 3: Put the result of the inner
integral back in and finish**

$$\int_1^2 \int_0^3 x^2 y \, dx dy = \int_1^2 g(x) \, dx = \int_1^2 \frac{9}{2} x^2 \, dx = \left| \frac{9}{2 \cdot 3} x^3 \right|_1^2 = \frac{9}{2 \cdot 3} (2^3 - 1^3) = \frac{63}{6}$$

Which is the same thing!

Let's see why switching the order of integration is so valuable

Example: $\int_1^2 \int_0^\pi y \sin(xy) \, dy \, dx$

Realization 1: I don't want to integrate $\int_0^\pi y \sin(xy) \, dy$

Realization 2: Let's switch and see what we get for it

$$\begin{aligned} \int_1^2 \int_0^\pi y \sin(xy) \, dy \, dx &= \int_0^\pi \int_1^2 y \sin(xy) \, dx \, dy \\ &= \int_0^\pi \left[\int_1^2 y \sin(xy) \, dx \right] dy = \int_0^\pi \left[\left. y \left(\frac{-1}{y} \cos xy \right) \right|_1^2 \right] dy \\ &= \int_0^\pi \left[- \left. \cos xy \right|_1^2 \right] dy = \int_0^\pi [-\cos(2y) - (-\cos y)] dy \\ &= \int_0^\pi [-\cos(2y) + \cos y] dy = \left. \left[\frac{-1}{2} \sin(2y) + \sin y \right] \right|_0^\pi = 0 \end{aligned}$$

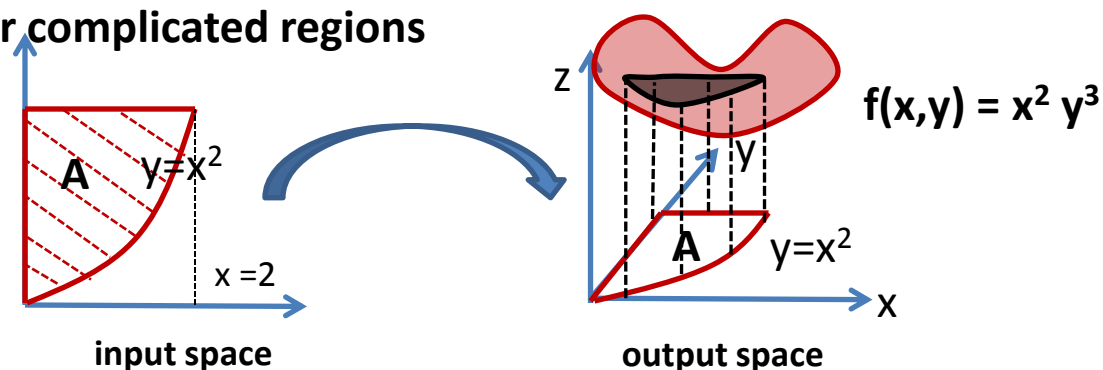
Section 15.2: Double Integrals over complicated regions

Consider $f(x,y) = x^2 y^3$

Find $\int \int_A x^2 y^3 dA$

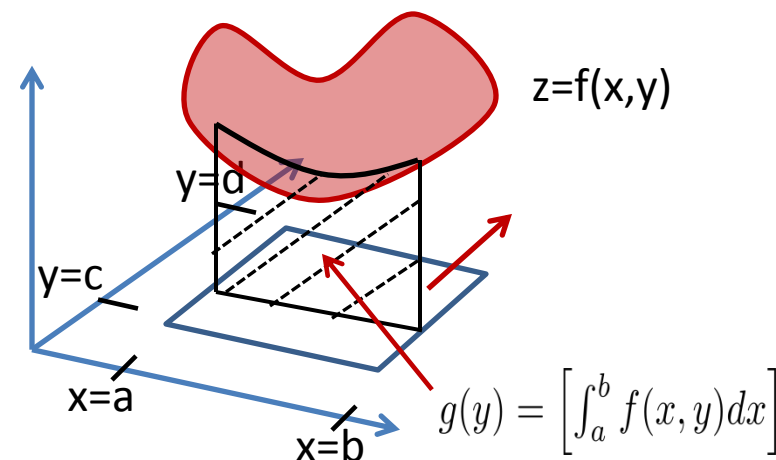
where A is the region shown in the figure

N.B. : Our answer is a **volume**



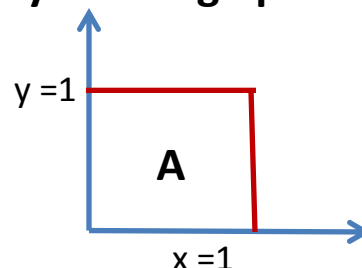
This is not as simple a problem as before—but we can do it.

The key idea will be to remember that a double integral is a sum of one-dimensional integrals—and each of those one-dimensional integrals can have **different limits of integration**



We will show how to do it by building up with an easier problem

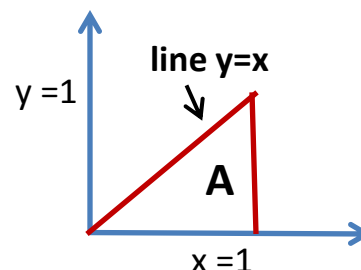
Question: what is the integral of $f(x,y)=2$ over the unit square?



$$\begin{aligned} \int_0^1 \int_0^1 f(x,y) \, dx \, dy &= \int_0^1 \int_0^1 2 \, dx \, dy \\ &= \int_0^1 \left[2x \right]_0^1 dy = \int_0^1 2 dy = 2 \end{aligned}$$

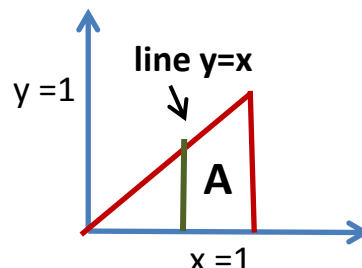
Which should be sorta obvious

Now what if I ask for the integral of $f(x,y)=2$ over A



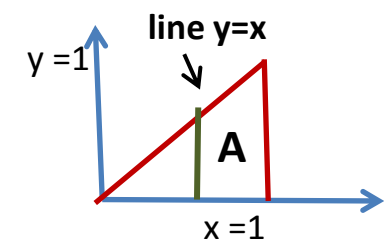
Should be obvious that the answer is half of above: so the answer should be **1**

But how do we set up the integral and get the limits?



- (1) x goes from 0 to 1
- (2) Given x , y goes from 0 to x
- (3) The contribution to the integral along the green line is $\int_0^x f(x,y) dy$

So, summing up over all the green line contributions: $\iint_A f(x,y) dA = \int_0^1 \left(\int_0^x f(x,y) dy \right) dx$



- (1) x goes from 0 to 1
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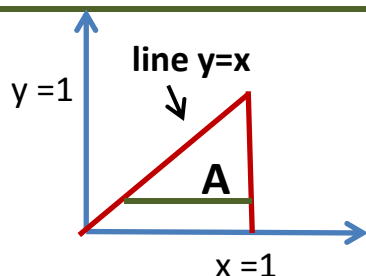
So, summing up over all the green line contributions: $\iint_A f(x, y) dA = \int_0^1 \left(\int_{y=0}^{y=x} f(x, y) dy \right) dx$

Let's do this integral and see if we get the expected value of **1**

$$\begin{aligned} \iint_A f(x, y) dA &= \int_0^1 \left(\int_0^x f(x, y) dy \right) dx = \int_0^1 \left(\int_0^x 2 dy \right) dx \\ &= \int_0^1 \left(\left[2y \right]_0^x \right) dx = \int_0^1 [2x - 2 \cdot 0] dx = \left[x^2 \right]_0^1 = (1)^2 - (0)^2 = 1 \end{aligned}$$

Yaay!

Finally, let's do the integral the other way:



- (1) y goes from 0 to 1
- (2) Given y , x goes from y to 1
- (3) The contribution to the integral along the green line is $\int_y^1 f(x, y) dx$

So, summing up over all the green line contributions: $\iint_A f(x, y) dA = \int_0^1 \left(\int_y^1 f(x, y) dx \right) dy$

Check that it is the same

$$\int_0^1 \left(\int_y^1 f(x, y) dx \right) dy = \int_0^1 \left(\int_y^1 2 dx \right) dy = \int_0^1 \left(\left[2x \right]_y^1 \right) dy = \int_0^1 (2 - 2y) dy = \left[2y - y^2 \right]_0^1 = 2 \cdot 1 - 1^2 = 1$$

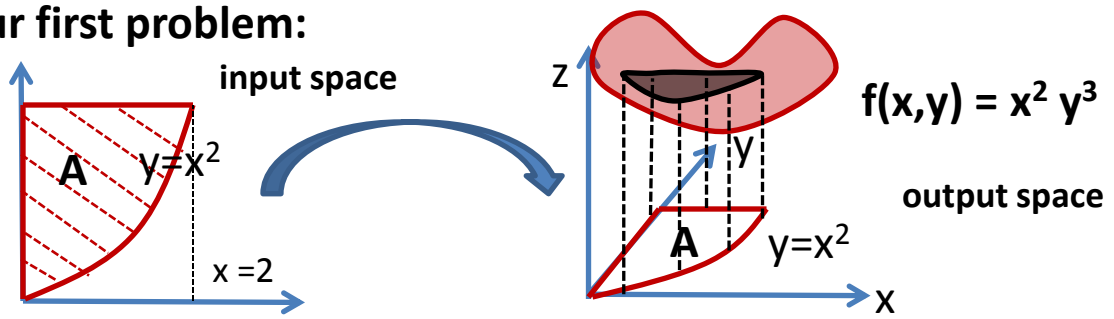
Yaay again!

Section 15.2: So let's go back to our first problem:

Consider $f(x,y) = x^2 y^3$

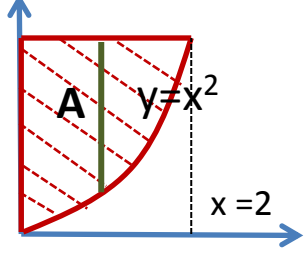
Find $\int \int_A x^2 y^3 dA$

where A is the region shown in the figure



Let's find the limits of integration: time for the green slices:

One way to go: x slices: So moving in y is the "inner integral"



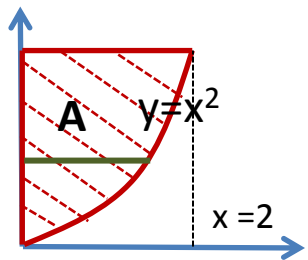
$$\int \int_A x^2 y^3 dA = \int_{\text{where } x \text{ starts}}^{\text{where } x \text{ ends}} \int_{\text{where } y \text{ starts}}^{\text{where } y \text{ ends}} x^2 y^3 dy dx$$

x goes from 0 to 2, so
$$= \int_0^2 \int_{\text{start of } y \text{ as a fcn. of } x}^{\text{end of } y \text{ as a fcn. of } x} x^2 y^3 dy dx$$

For a given x, y goes from x^2 to 4, so
$$= \int_0^2 \left[\int_{x^2}^4 x^2 y^3 dy \right] dx$$

inner integral (points to dy) outer integral (points to dx)

A different way to go: y slices: So moving in x is the inner integral



$$\int \int_A x^2 y^3 dA = \int_{\text{where } y \text{ starts}}^{\text{where } y \text{ ends}} \int_{\text{where } x \text{ starts}}^{\text{where } x \text{ ends}} x^2 y^3 dx dy$$

y goes from 0 to 4, so
$$= \int_0^4 \int_{\text{start of } x \text{ as a fcn. of } y}^{\text{end of } x \text{ as a fcn. of } y} x^2 y^3 dx dy$$

For a given y, x goes from 0 to sqrt(y), so
$$= \int_0^4 \left[\int_0^{\sqrt{y}} x^2 y^3 dx \right] dy$$

inner integral (points to dx) outer integral (points to dy)

One way to go: Slicing in x: So moving in y is the inner integral

$$\begin{aligned}
 &= \int_0^2 \left[\int_{x^2}^4 x^2 y^3 \, dy \right] dx = \int_0^2 \left[\left. \frac{x^2 y^4}{4} \right|_{x^2}^4 \right] dx = \int_0^2 \left[\frac{x^2 (4)^4}{4} - \frac{x^2 (x^2)^4}{4} \right] dx \\
 &= \int_0^2 \left[64x^2 - \frac{x^{10}}{4} \right] dx = \left. \left[\frac{64x^3}{3} - \frac{x^{11}}{44} \right] \right|_0^2 = \frac{512}{3} - \frac{2048}{44} = \frac{16384}{132}
 \end{aligned}$$

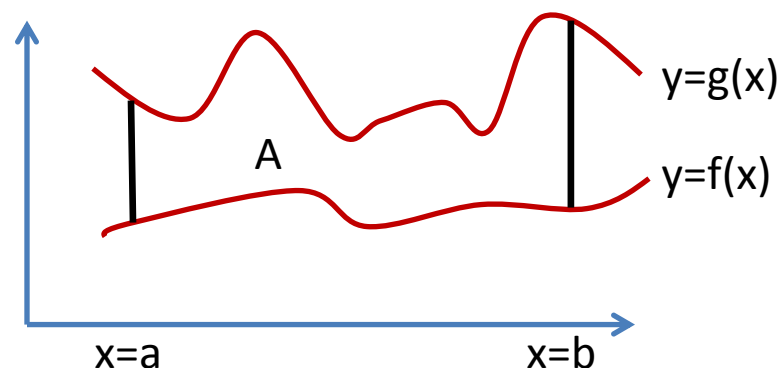
A different way to go: Slicing in y: So moving in x is the inner integral

$$\begin{aligned}
 &= \int_0^4 \left[\int_0^{\sqrt{y}} x^2 y^3 \, dx \right] dy = \int_0^4 \left[\left. \frac{x^3 y^3}{3} \right|_0^{\sqrt{y}} \right] dy = \int_0^4 \frac{1}{3} y^{\frac{9}{2}} dy = \left. \frac{2}{33} y^{\frac{11}{2}} \right|_0^4 = \frac{2}{33} 2^{11} \\
 &= \frac{16384}{132}
 \end{aligned}$$

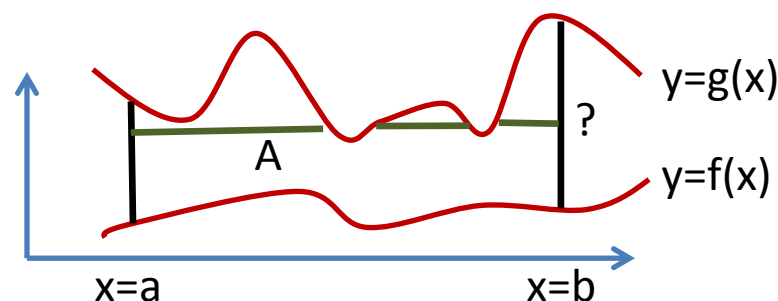
Same thing

The trick with these is to think before slicing

Set up the integral of $w(x,y) = \sin(xy)$ over the region A



Inner slicing in x would be a bad idea!!!
(breaks into too many pieces)



Instead, inner slicing in y

$$\int_a^b \int_{f(x)}^{g(x)} w(x,y) dy dx = \int_a^b \left[\int_{f(x)}^{g(x)} \sin xy dy \right] dx$$

