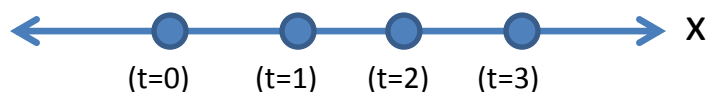


Review of First Third of the Course

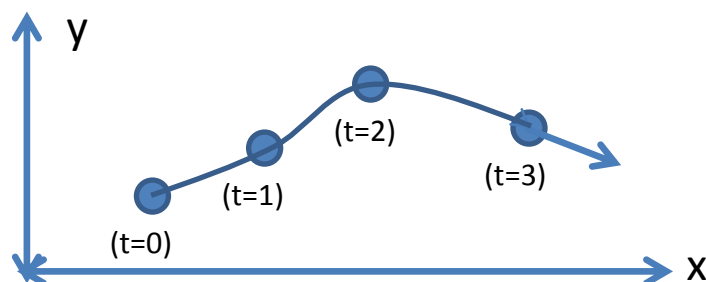
Parameterized curves



$x = f(t)$ 1D parameterized function

Mapping from 1D to 1D

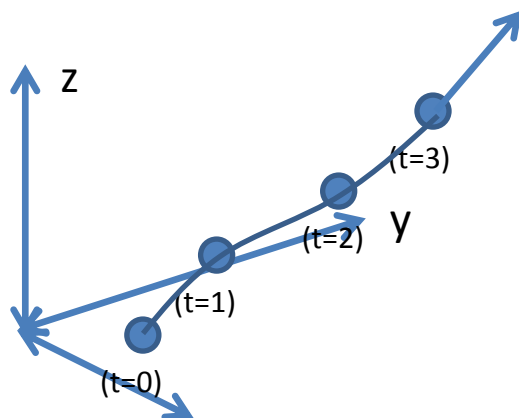
Gives x coordinate of object moving in 1D at time t .



$x = f(t), y = g(t)$ 1D parameterized function

Mapping from 1D to 2D

Gives (x, y) coordinate of object moving in 2D at time t .



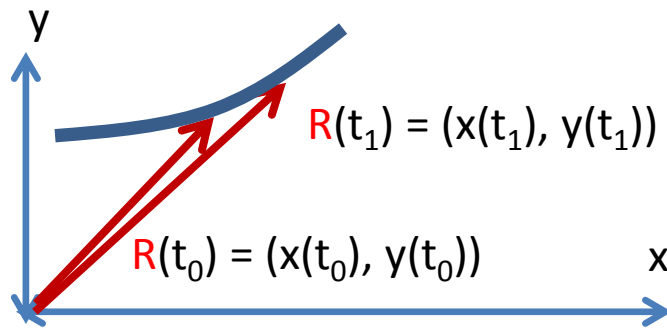
$x = f(t), y = g(t), z = h(t)$ 1D parameterized function

Mapping from 1D to 3D

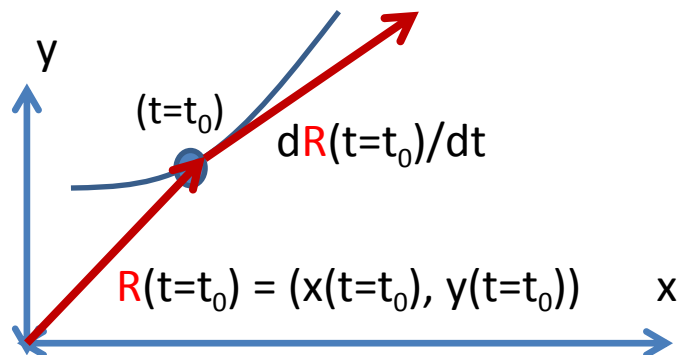
Gives (x, y, z) coordinate of object moving in 3D at time t .

Review of First Third of the Course

Derivatives of parameterized curves



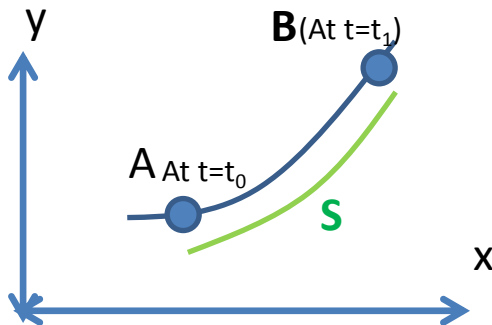
$R(t) = (x(t), y(t))$ is vector description of curve in 2D.



$dR(t)/dt = (dx(t)/dt, dy(t)/dt)$ is vector derivative of particle trajectory
It is tangent to the curve at the point $x(t=t_0), y(t=t_0)$

Review of First Third of the Course

Arc-length of parameterized curves



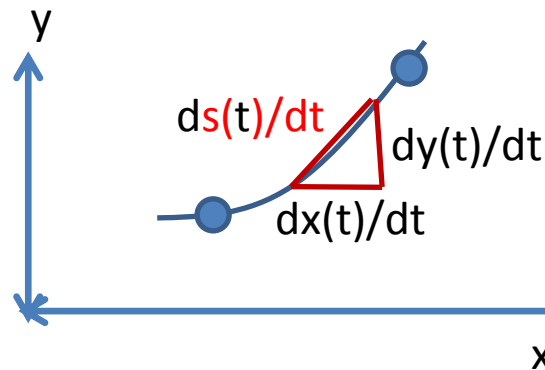
$(x(t), y(t))$ describes curve in 2D.
What is the total arc-length **S** from A to B?

$dS(t)/dt = [(dx(t)/dt)^2 + (dy(t)/dt)^2]^{(1/2)}$ using “bob’s” theorem.

Add them all up to get total arc-length

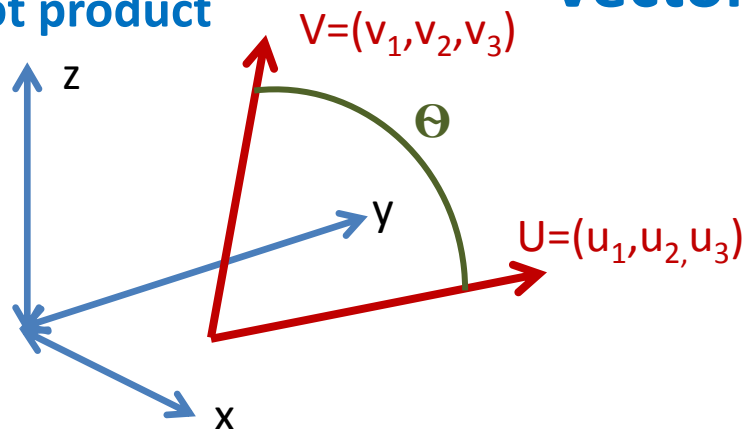
$$S = \int_{t_0}^{t_1} [ds(t)/dt] dt$$

$$S = \int_{t_0}^{t_1} [(dx(t)/dt)^2 + (dy(t)/dt)^2]^{(1/2)} dt$$



Dot product

Vectors



Two vectors in 3D: angle between is Θ

Definition of dot product:

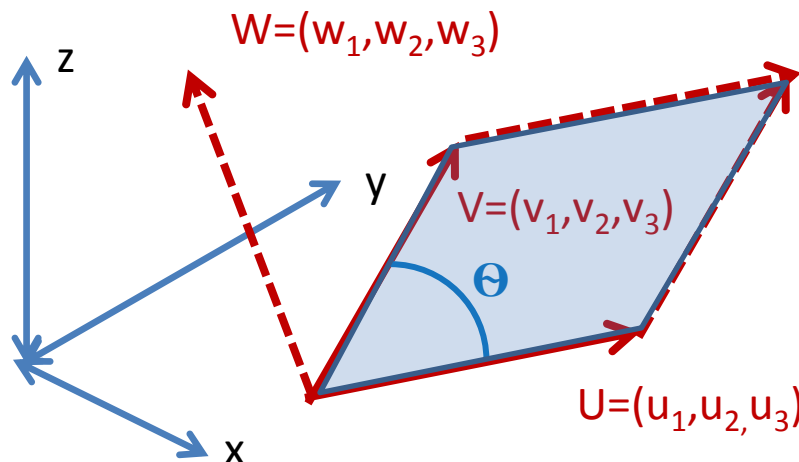
$$\mathbf{U} \cdot \mathbf{V} = (u_1 * v_1 + u_2 * v_2 + u_3 * v_3)$$

Proved: $\mathbf{U} \cdot \mathbf{V} = |\mathbf{U}| |\mathbf{V}| \cos \Theta$

Proved:

Two vectors are orthogonal $\iff \mathbf{U} \cdot \mathbf{V} = 0$

Cross product



Given two vectors in 3 dimensions

Can define their cross product as $\mathbf{U} \times \mathbf{V} =$

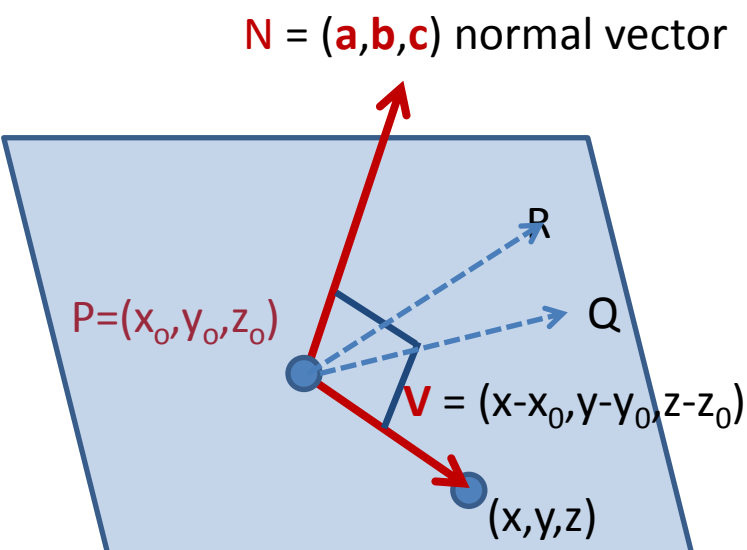
$$\mathbf{W} = (u_2 * v_3 - u_3 * v_2, u_3 * v_1 - u_1 * v_3, u_1 * v_2 - u_2 * v_1)$$

Proved: $\mathbf{W} = \mathbf{U} \times \mathbf{V}$ is orthogonal to $\mathbf{U}$ and to $\mathbf{V}$

Proved: $|\mathbf{U} \times \mathbf{V}| = |\mathbf{U}| |\mathbf{V}| \sin \Theta$

Proved: $|\mathbf{U} \times \mathbf{V}| = \text{area of parallelogram}$

Defining planes using vectors



A plane is completely described if you are given a point $P = (x_0, y_0, z_0)$ and normal vector N

For any point (x, y, z) in the plane, the vector that connects (x, y, z) to P is given by $V = (x - x_0, y - y_0, z - z_0)$

We can write an equation for any point (x, y, z) in the plane by realizing that any such vector V in the plane must be orthogonal to N : that is, $N \cdot V = 0$

So any point (x, y, z) in the plane must satisfy $N \cdot V = 0$: which is the same as $(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0$ which becomes $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

If we are not given a normal vector N , but instead have three points P , Q , and R in the plane, we can find a normal vector N by taking the cross product $N = (Q - P) \times (R - P)$

Functions of more than one variable

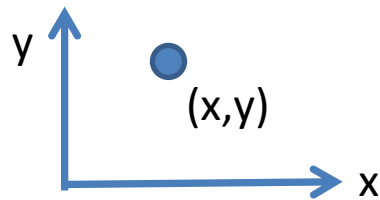
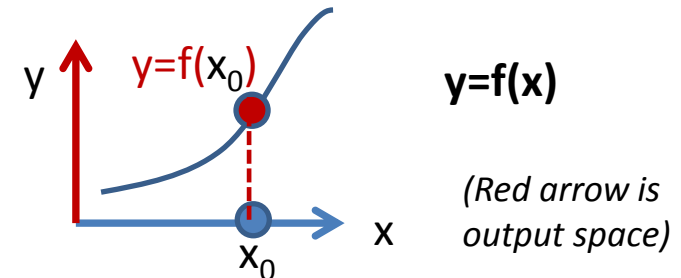
Input space

Need an extra dimension to graph it



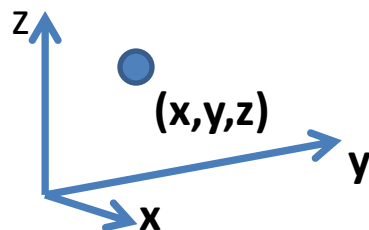
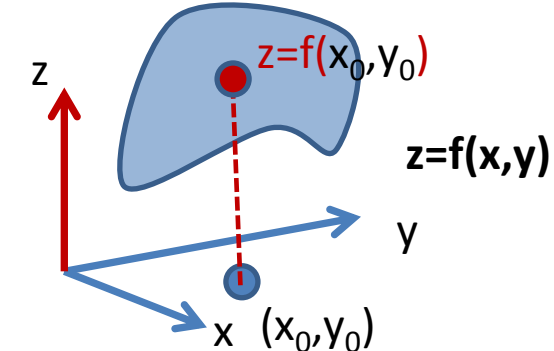
$f(x)$: 1D input
to 1D output

Function of one variable



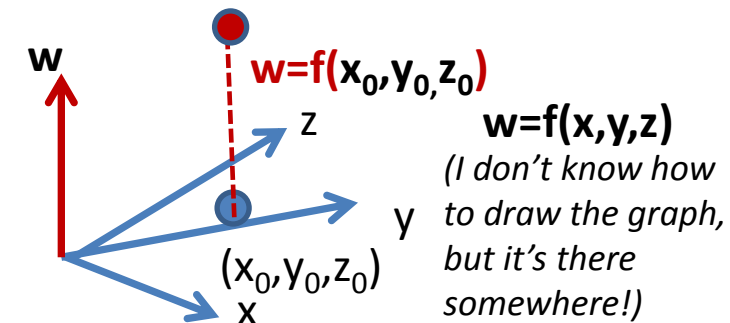
$f(x,y)$: 2D input
to 1D output

Function of two variables

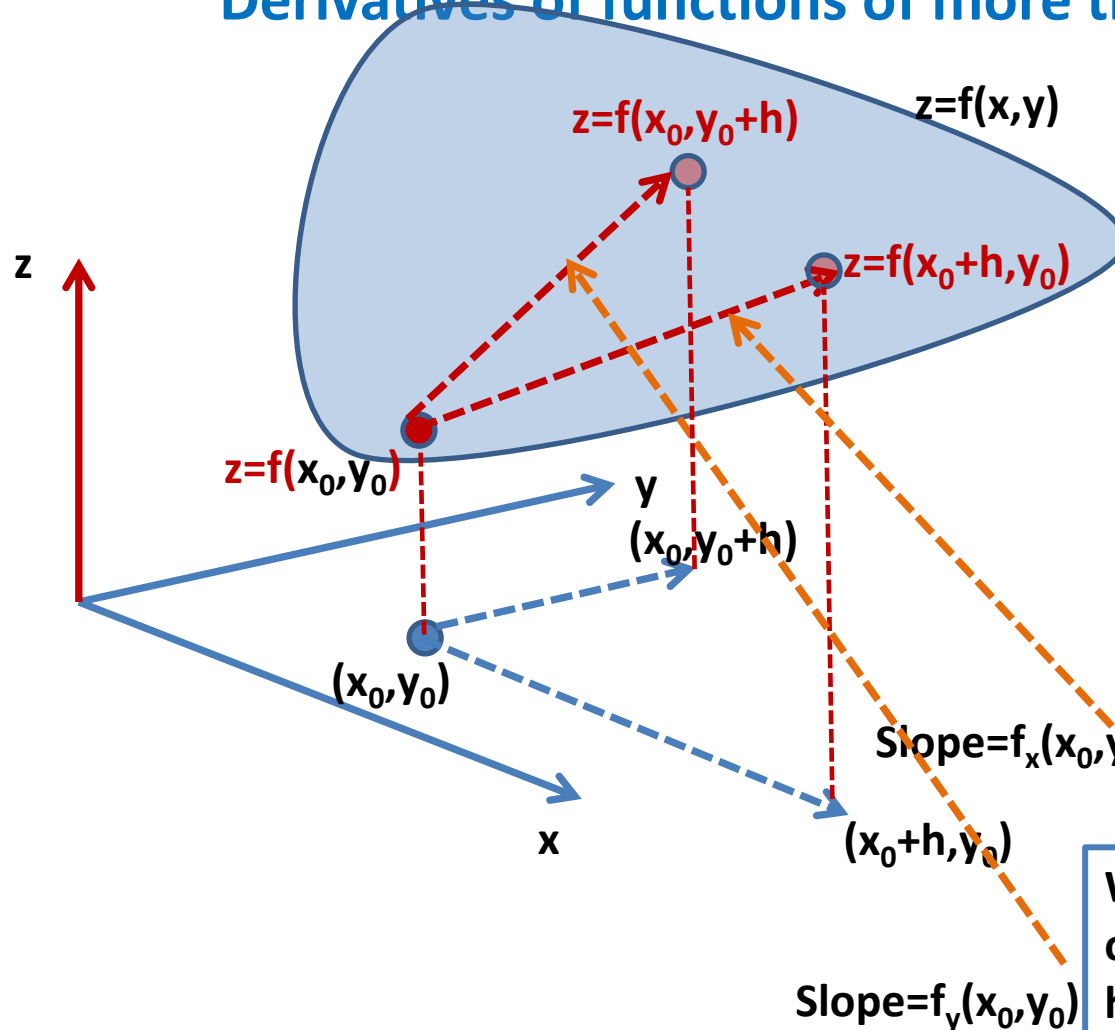


$f(x,y,z)$: 3D input
to 1D output

Function of three variables



Derivatives of functions of more than one variable



Start with a function $f(x, y)$
The graph is 2D surface in $\mathbb{R}^3$

At an input point (x_0, y_0) the function has an output $f(x_0, y_0)$

We can ask “how does the output change as x increases, holding y fixed?”

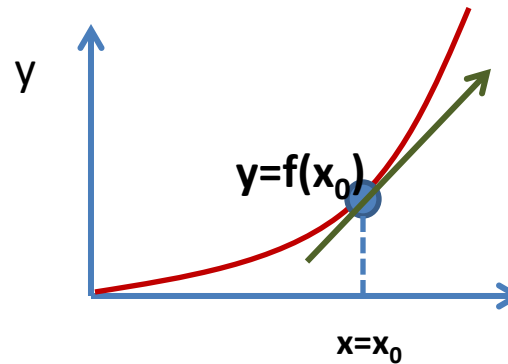
The slope of the line connecting these changes is $f_x(x_0, y_0)$

We can also ask “how does the output change as y increases, holding x fixed?” Answer= $f_y(x_0, y_0)$

Section 14.4: Tangent Planes: **WHEN YOU ARE GRAPHING OUTPUT AGAINST INPUT!!!!**

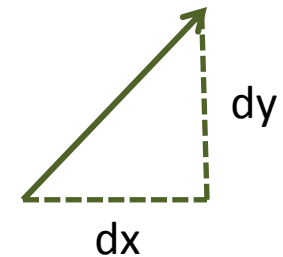
Let's recall 1D Calculus: $y=f(x)$

Tangent line at $(x_0, f(x_0))$ touches the graph $y=f(x)$ at only one point near $(x_0, f(x_0))$



Tangent line with slope $f'(x_0)$ going through the point $(x_0, f(x_0))$

The tangent line is given by $y - f(x_0) = \text{slope}(x - x_0) = \left. \frac{df}{dx} \right|_{x_0} (x - x_0)$

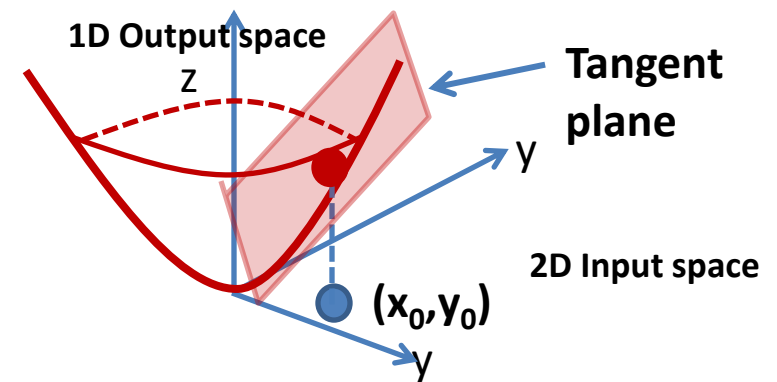


Tangent vector = $(dx, dy) = (1, dy/dx) = (1, f'(a))$

We want to construct a similar idea for functions of two (or more variables):

The Tangent Plane

Tangent plane at $(x_0, y_0, f(x_0, y_0))$ touches the graph $z=f(x, y)$ at only one point near $(x_0, y_0, f(x_0, y_0))$



Section 14.4: Tangent Planes: WHEN YOU ARE GRAPHING OUTPUT AGAINST INPUT!!!!**1d: Slope of a tangent line**

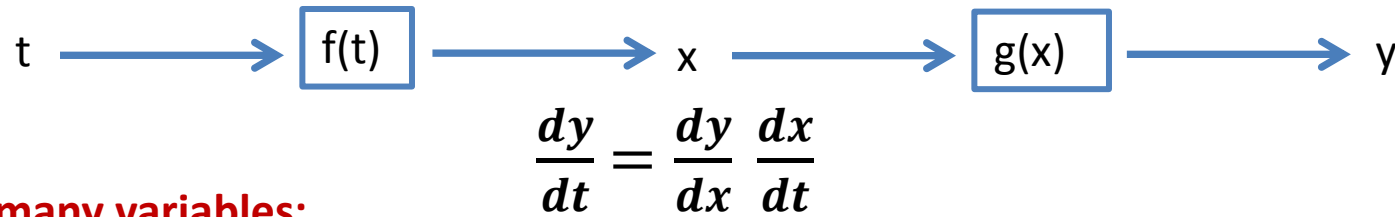
$$f(x) = f(a) + \left. \frac{df}{dx} \right|_a (x - a) \leftarrow \text{Equation for Line tangent to } a, f(a)$$

2d: Formula for the tangent plane

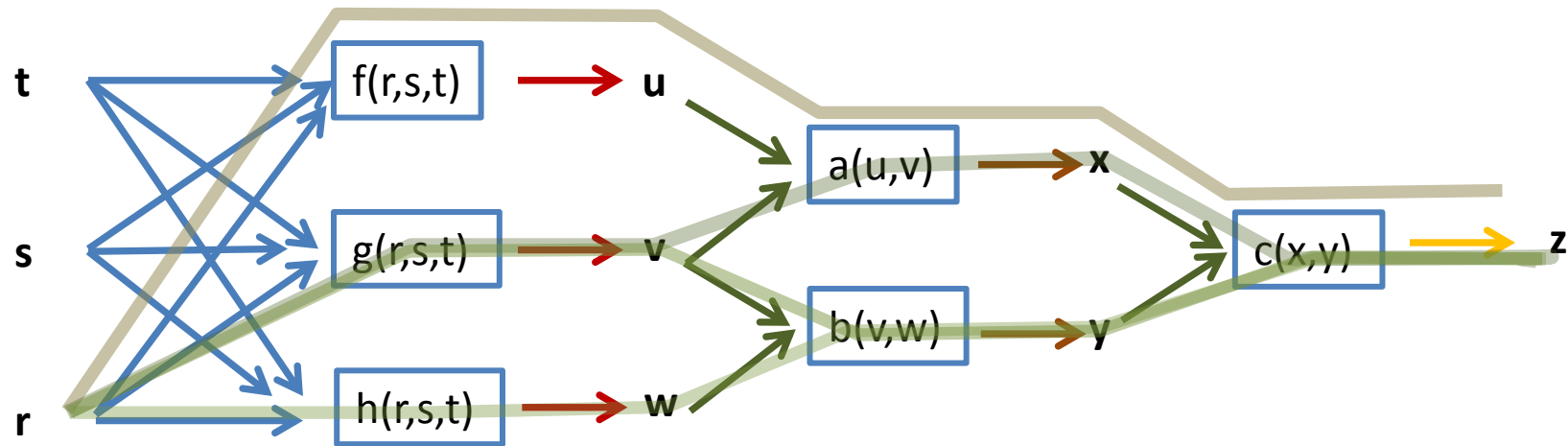
$$f(x, y) = f(a, b) + \left. \frac{\partial f}{\partial x} \right|_{a,b} (x - a) + \left. \frac{\partial f}{\partial y} \right|_{a,b} (y - b) \leftarrow \text{Equation for plane tangent to } a, b, (a, b)$$

The Multi-Dimensional Chain Rule

Of one variable:



Of many variables:



To find $\frac{\partial z}{\partial r}$ follow all pathways through the graph:

$$\frac{\partial z}{\partial r} =$$

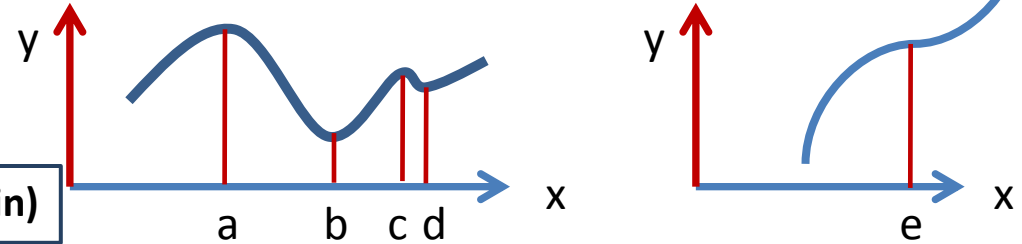
Critical Values of Multivariable functions

One dimension $y=f(x)$

Critical points: Places where $df/dx = 0$

If, in addition, $d^2f/dx^2 < 0$ (> 0) then max (min)

$x=a$: global max || $x=b$: global min || $x=a,c$: local maximum || $x=b,d$: local min || $x=e$: (inflection)



Two dimensions $z=f(x,y)$

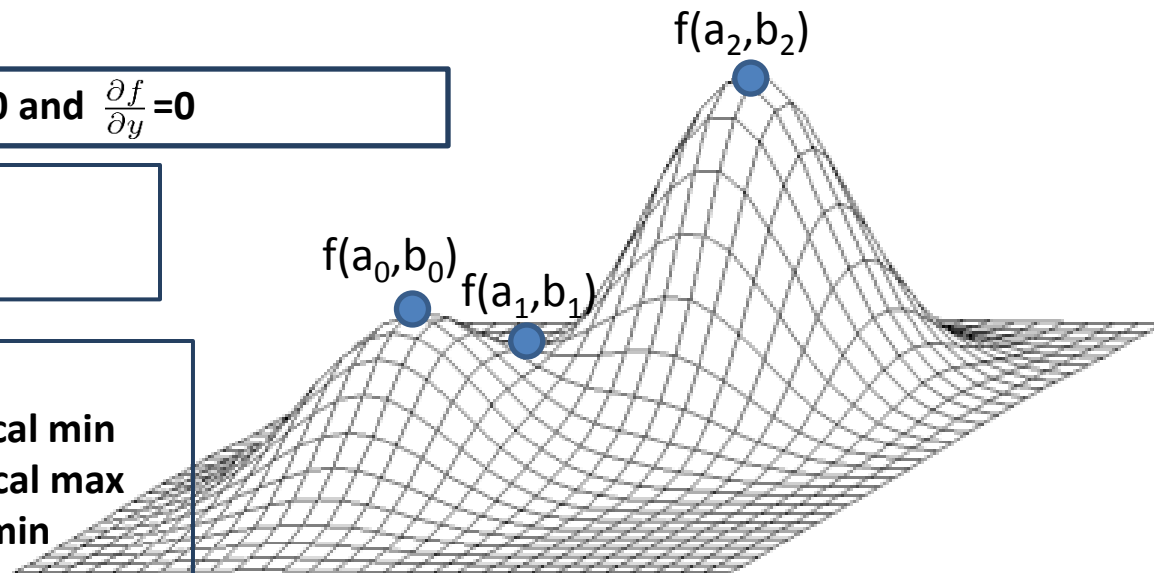
Critical points: Places where both $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

At the input point (a,b) define D as

$$D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

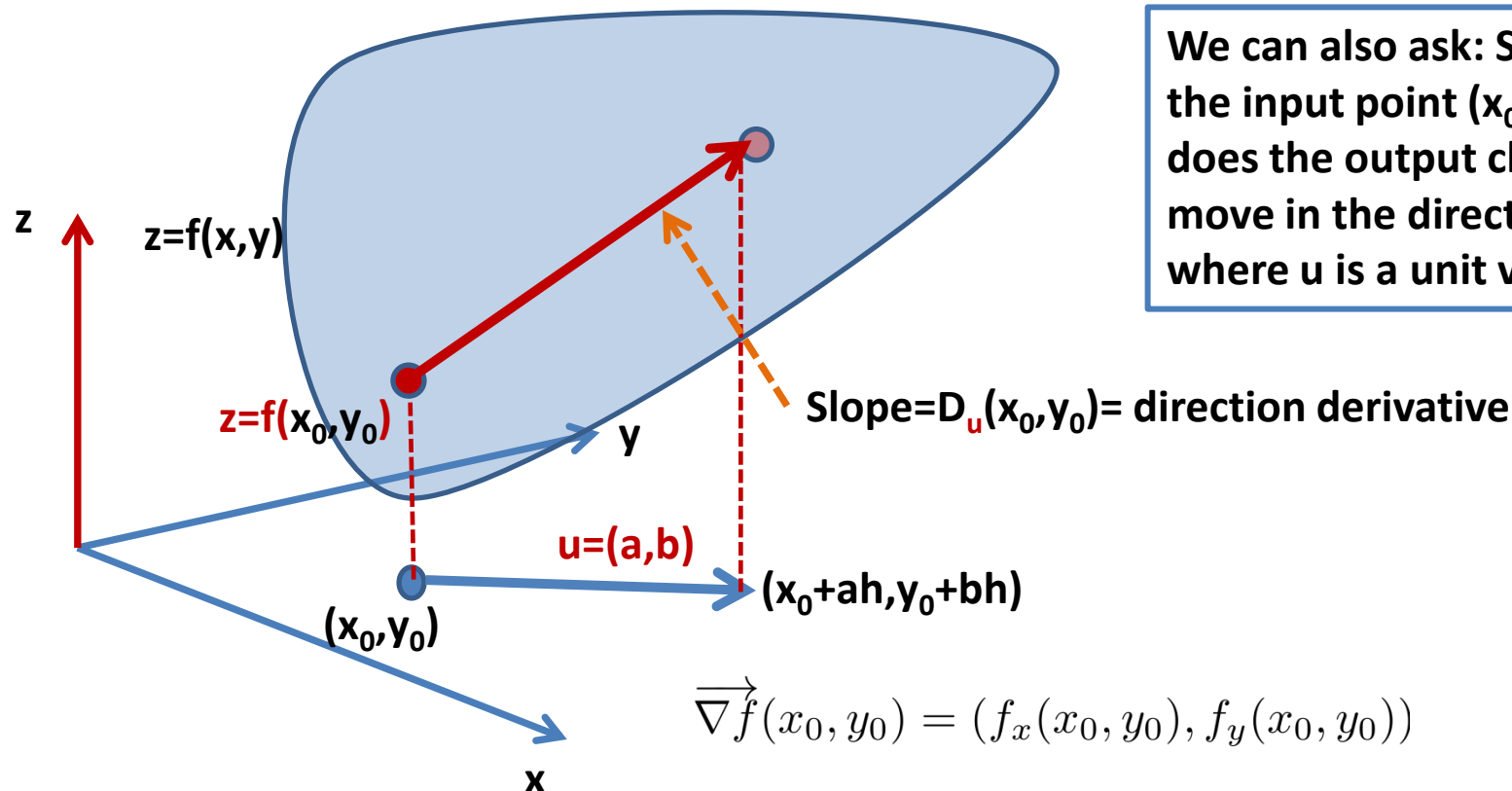
Suppose (a,b) is a critical point, then

- (a) if $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then local min
- (b) if $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then local max
- (c) if $D(a,b) < 0$, then neither max nor min



(a_0, b_0) = local max. (a_1, b_1) = neither (saddle) (a_2, b_2) = global max

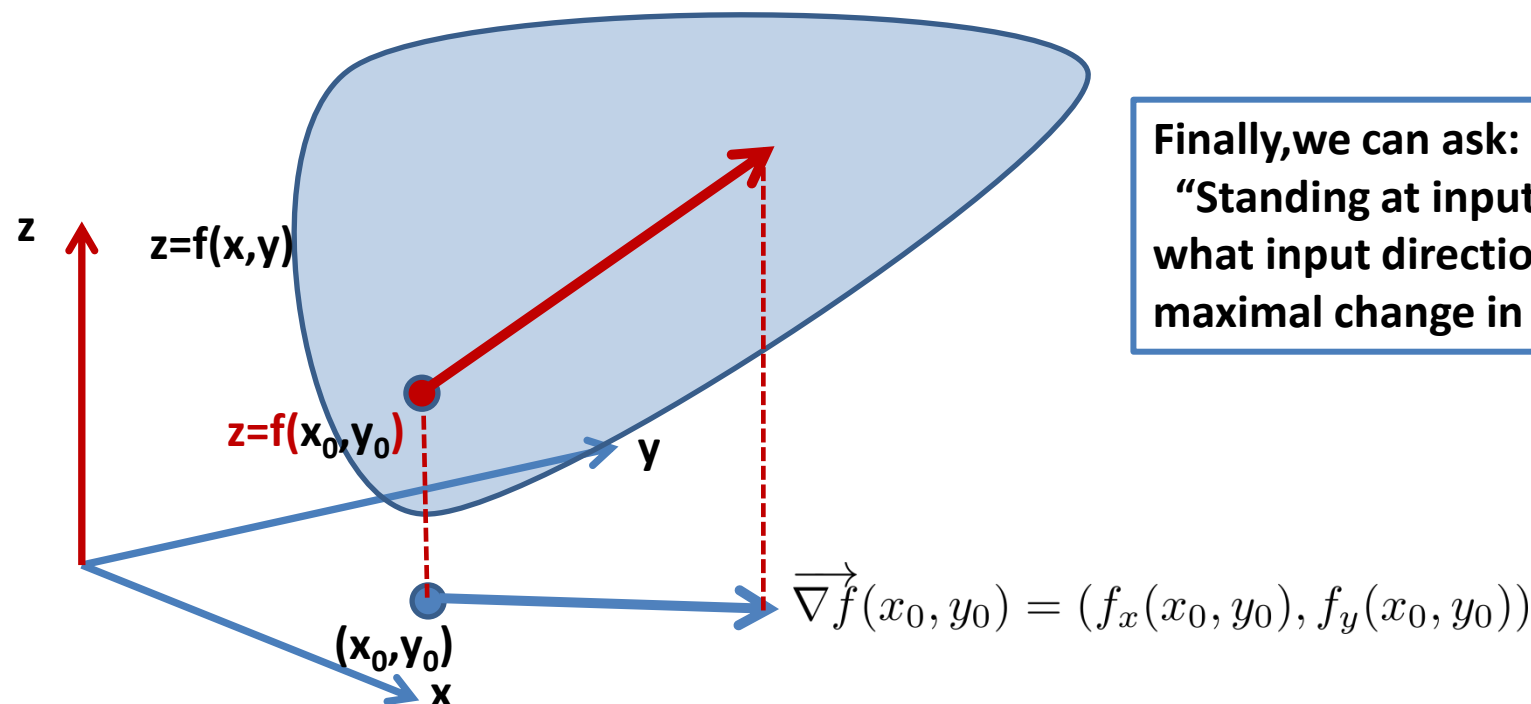
Direction Derivatives



And we proved that:

$$D_{\mathbf{u}}(x_0, y_0) = a * f_x(x_0, y_0) + b * f_y(x_0, y_0) = \vec{\nabla} f \cdot \vec{u} \quad \text{At } x_0, y_0$$

The gradient



Finally, we can ask:

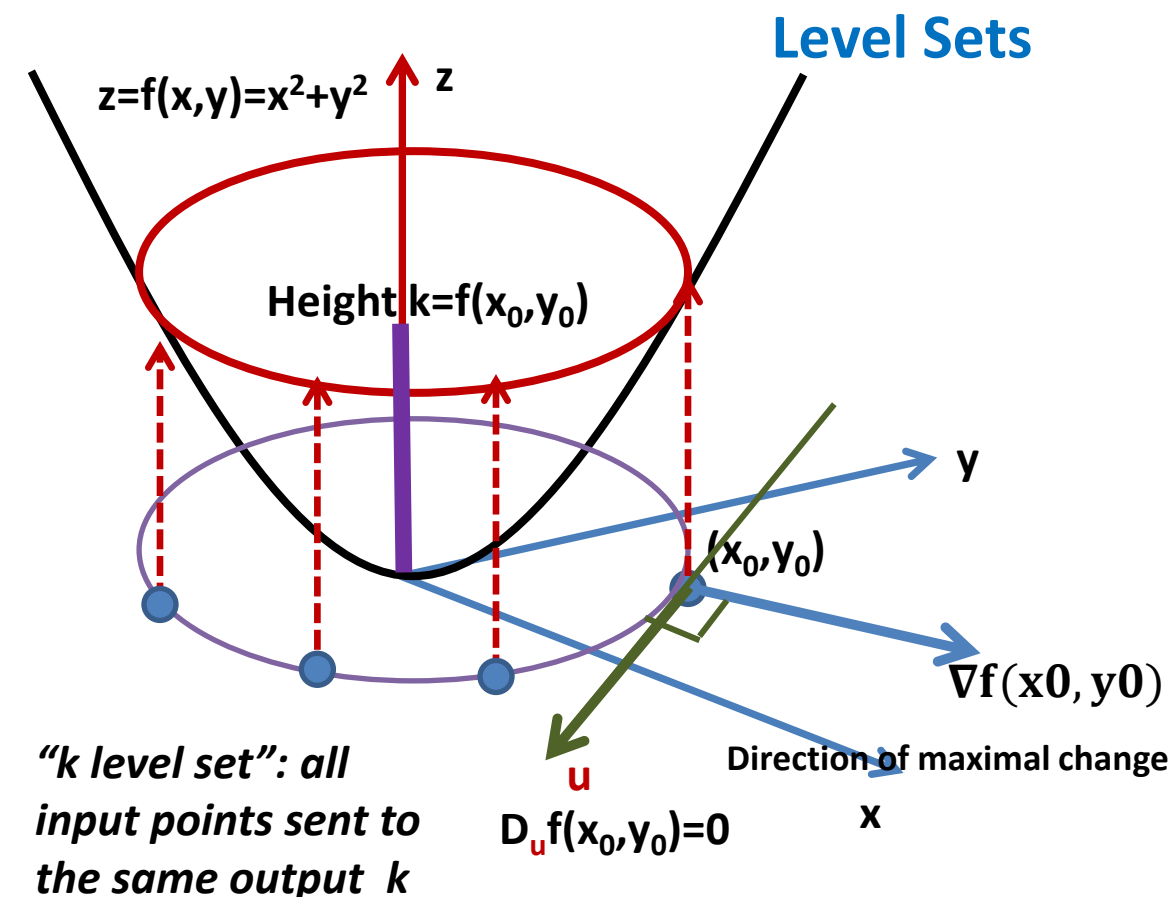
"Standing at input point (x_0, y_0) , what input direction gives the maximal change in $f(x, y)$?"

The answer is to move in the gradient direction

$$\vec{\nabla} f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$$

Why? Because $D_{\vec{u}} f(x_0, y_0) = \vec{\nabla} f(x_0, y_0) \cdot \vec{u} = |\vec{\nabla} f| |\vec{u}| \cos \theta$ **is biggest when** $\cos \theta = 1$

And when $\cos \theta = 1$, then $\theta = 0$ so $\vec{\nabla} f$ and $\vec{u}$ point in the same direction



Start with a function $f(x, y)$
The graph is 2D surface in $\mathbb{R}^3$

Level set with value k is the set of all input points sent to the value k .

Moving along the k level set in input space, the function $f(x, y)$ doesn't change

So, if u is a direction along the k level set, then $D_u f(x_0, y_0) = 0$

The gradient at $f(x_0, y_0)$ is the direction of maximal change

Since $D_u f(x_0, y_0) = 0$ and since $0 = D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot u$

Then $u(x_0, y_0)$ is perpendicular to $\nabla f(x_0, y_0)$

So the gradient $\nabla f(x_0, y_0)$ is normal to the tangent to the level set going through (x_0, y_0)

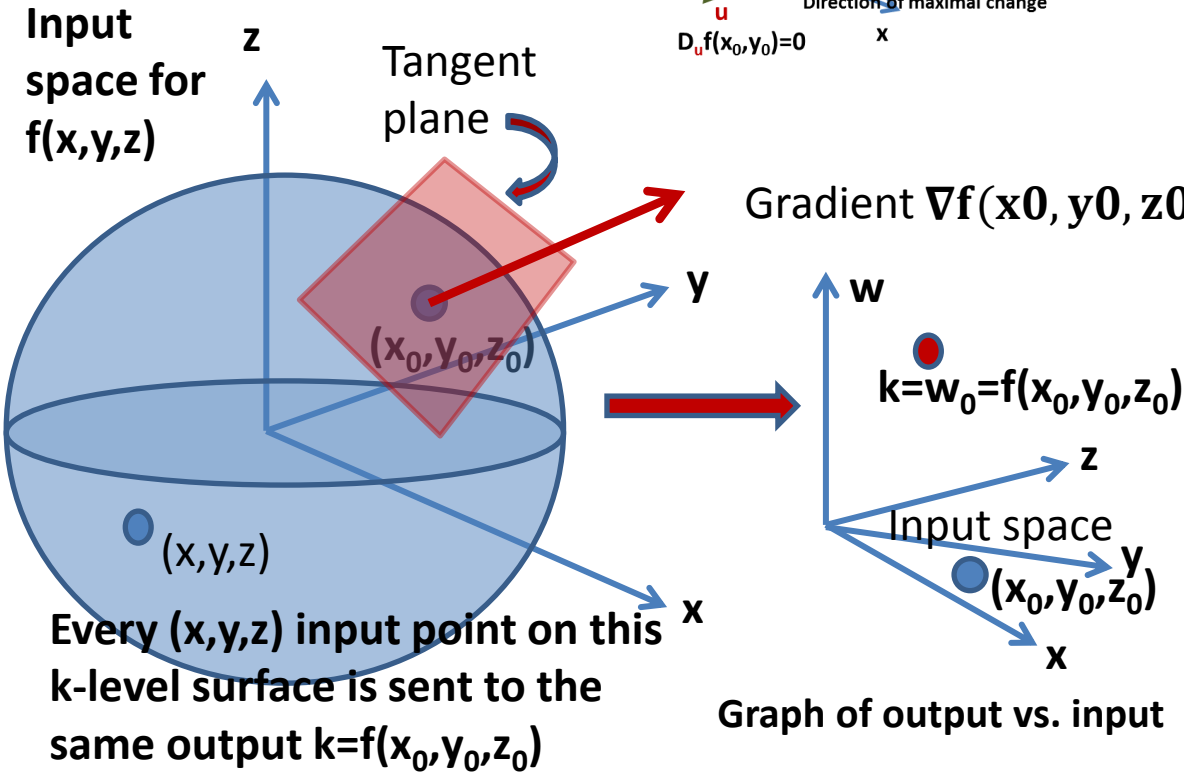
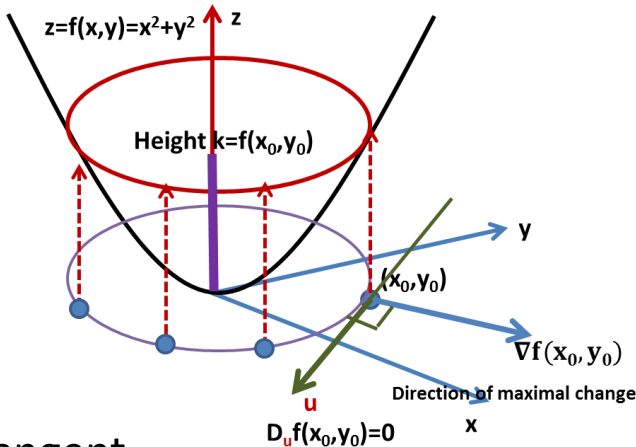
Level Sets in Higher Dimensions

Again. At input (x_0,y_0) the gradient is normal to the line tangent to the $k=f(x_0,y_0)$ level curve passing through (x_0,y_0) .

This same idea is true in higher dimensions. For a function $w=f(x,y,z)$, the input space is three-dimensional.

At input (x_0,y_0,z_0) , the gradient is normal to the *plane tangent* to the $k=f(x_0,y_0,z_0)$ level set passing through (x_0,y_0,z_0) .

So we have an equation for the tangent plane at the point (x_0,y_0,z_0) , since we know a point and the normal $\nabla f(x_0,y_0,z_0)$



Finally, there is some (understandable) confusion about tangent planes. Let me try to sort it out

Suppose $y=f(x)$. Find the tangent at input x_0 **Example: $y=x^2$ $x_0 = 3$** ***Two* ways to do this:**

Way #1: Interpret $y=f(x)$ as the graph of output against input:

A mapping from $\mathbb{R}^1 \rightarrow \mathbb{R}^1$

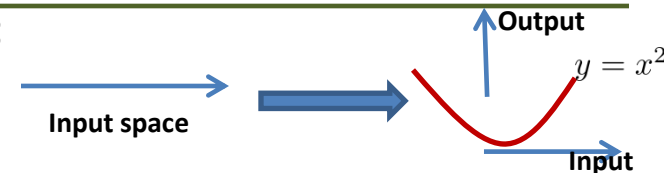
Step 1: Draw the output against input

Step 2: Want the tangent point at $(3, f(3)) = (3, 9)$

Step 3: We need the slope to use the formula $(y - y_0) = (slope) * (x - x_0)$

Step 4: Slope is $= df/dx = 2x$: at input $x_0=3$, slope is 6.

So answer is $y-9 = 6(x-3)$ which we can rewrite as $y = 6x - 9$



Way #2: Consider a new function $w(x,y) = y-f(x)$: $w(x,y)=y-x^2$ is a mapping from $\mathbb{R}^2 \rightarrow \mathbb{R}^1$

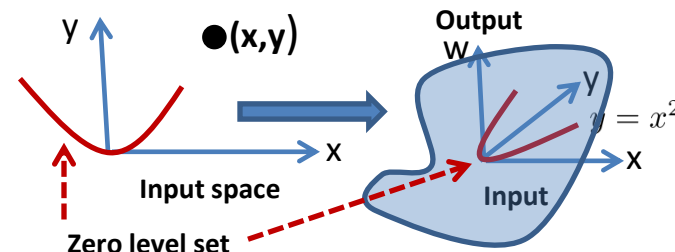
Step 1: Realize that the zero level set of function $w(x,y)$

is the red curve, which is the set of all (x,y) sent to output 0, in other words, whenever $y=f(x)$

Step 2: So, the input $x_0=3$, $y_0=9$ gets sent to $w = 0$

Step 3: That means that at the input $(3,9)$ the gradient $\nabla w(x,y) = (-2x, 1) = (-6, 1)$ is normal to the tangent to the level set $0=y-x^2$

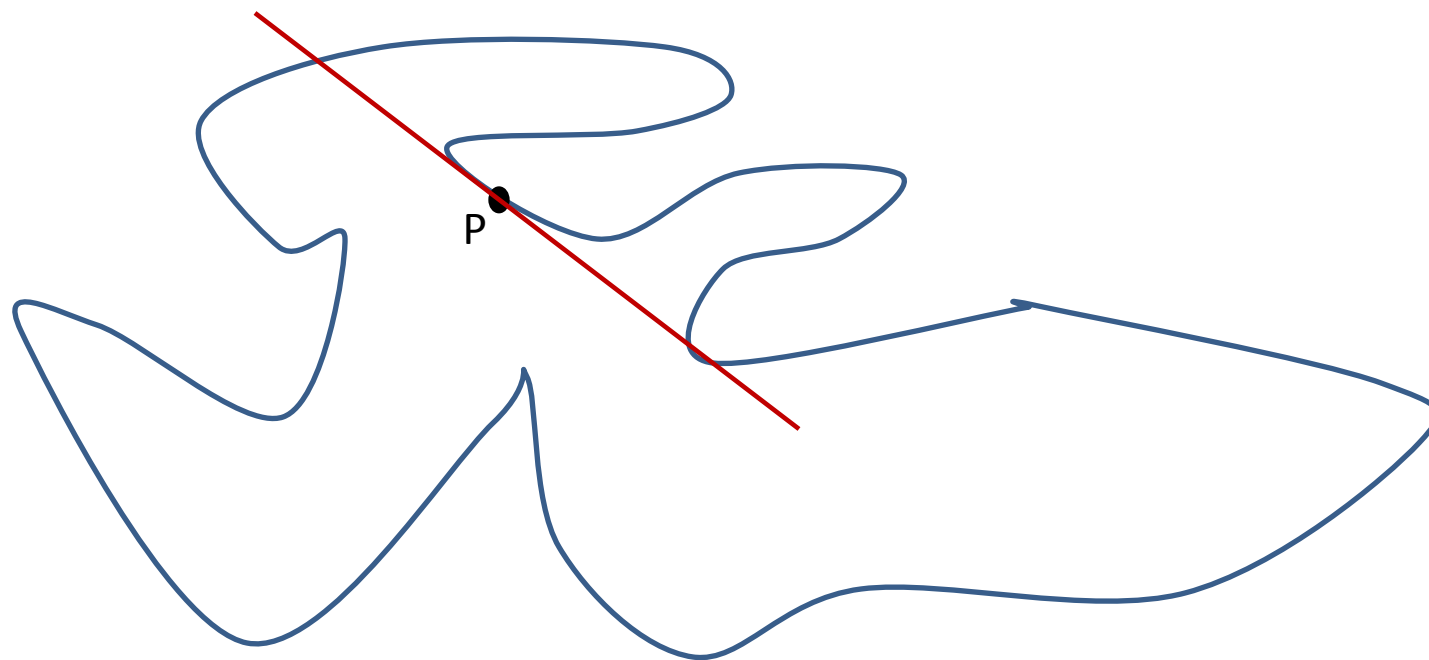
Step 4: So, using the normal equation for a line $\vec{n} \cdot (x - x_0, y - y_0) = 0$



$$(-6, 1) \cdot (x - 3, y - 9) = 0 \rightarrow -6x + 18 + y - 9 = 0 \rightarrow y = 6x - 9$$

Final question: Why would you ever use the second way?

Answer: because you might get asked to find the tangent to the following funky curve at point P –it's not a function....



Example: Find the tangent to $w=f(x,y,z) = x^2 + y^2 + z^2$ at the input $(1,2,3)$

Way #1: Interpret $w=f(x,y,z)$ as the graph of output against input: A mapping from $\mathbb{R}^3 \rightarrow \mathbb{R}^1$

Step 1: Want the tangent object at the point $(1,2,3,f(1,2,3)) = (1,2,3,14)$

Step 2: We need the various partials to use the formula

$$f(x, y, z) = f(a, b, c) + \left. \frac{\partial f}{\partial x} \right|_{a,b,c} (x - a) + \left. \frac{\partial f}{\partial y} \right|_{a,b,c} (y - b) + \left. \frac{\partial f}{\partial z} \right|_{a,b,c} (z - c)$$

Equation for hyperplane tangent to $f(a, b, c)$

Step 4: $f_x=2x, f_y=2y, f_z=2z$ ---so $(f_x, f_y, f_z) = (2, 4, 6)$.

So tangent object is $w=14 + (2)(x-1)+(4)(y-2)+(6)(z-3) = 2x+4y+6z+14$

Way #2: Think of the equation $w=f(x,y,z)$ as a particular level set of a new function

$U(x,y,z,w) = w-f(x,y,z)=w-(x^2+y^2+z^2)$ is a mapping from $\mathbb{R}^4 \rightarrow \mathbb{R}^1$

Step 1: Realize that the zero level set of function $U(x,y,z,w)$

is the set of all (x,y,z,w) sent to output 0

Step 2: So, the input $x_0=1, y_0=2, z_0=3, w_0=14$ gets sent to $U = 0$

Step 3: That means that at the input $(1,2,3,14)$ the gradient

is normal to the tangent to the level set $0= w-(x^2+y^2+z^2)$

Step 4: The gradient at input $(1,2,3,14)$ is

$$\nabla U(x, y, z, w) = (-2x, -2y, -2z, 1) = (-2, -4, -6, 1) = \vec{n}$$

$$\vec{n} \cdot (x - x_0, y - y_0, z - z_0, w - w_0) = 0$$

$$(-2, -4, -6, 1) \cdot (x - 1, y - 2, z - 3, w - 14) = 0 \rightarrow w = 2x + 4y + 6z + 14$$