

Review of last lecture:

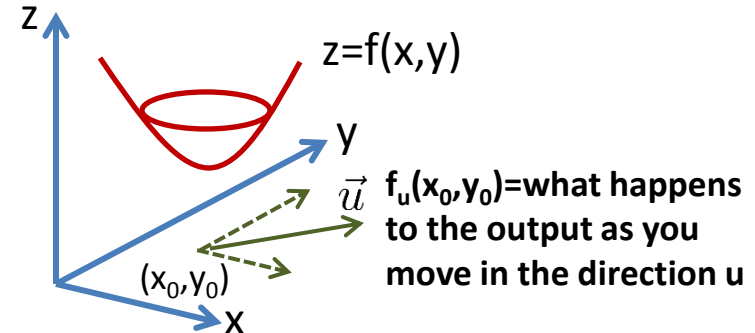
(1) Definition: $D_{\vec{u}}f(x_0, y_0) \equiv \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$

where $\vec{u}$ is a unit vector with components

$$\vec{u} = (a, b) \quad |\vec{u}| = 1$$

(2) And we found that:

$$D_{\vec{u}}f(x, y) = \vec{\nabla} f \cdot \vec{u}$$



(3) What direction makes $D_{\vec{u}}f(x, y) = \vec{\nabla} f \cdot \vec{u}$ the biggest?

$$\text{Since } D_{\vec{u}}f(x, y) = \vec{\nabla} f \cdot \vec{u} = |\vec{\nabla} f| |\vec{u}| \cos \theta$$

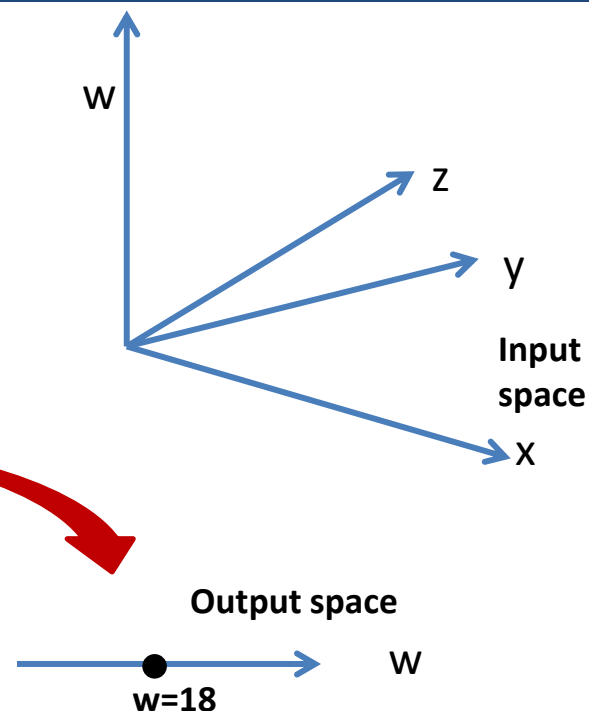
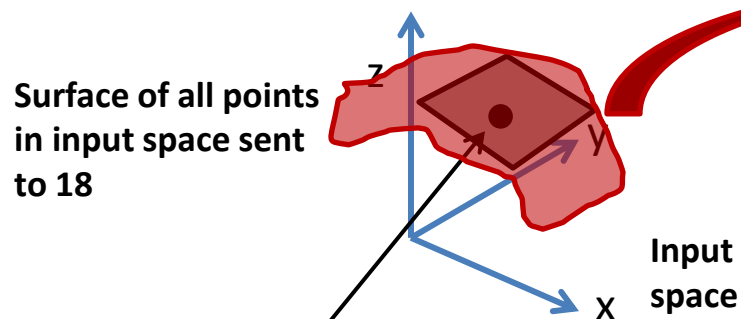
This is biggest when $\cos \theta = 1$ which happens when

$$\vec{\nabla} f \quad \text{and} \quad \vec{u} \quad \textbf{point in the same direction!}$$

Let's use these new ideas to think about tangent planes again

$$D_u f(x, y) = \vec{\nabla} f \cdot \vec{u}$$

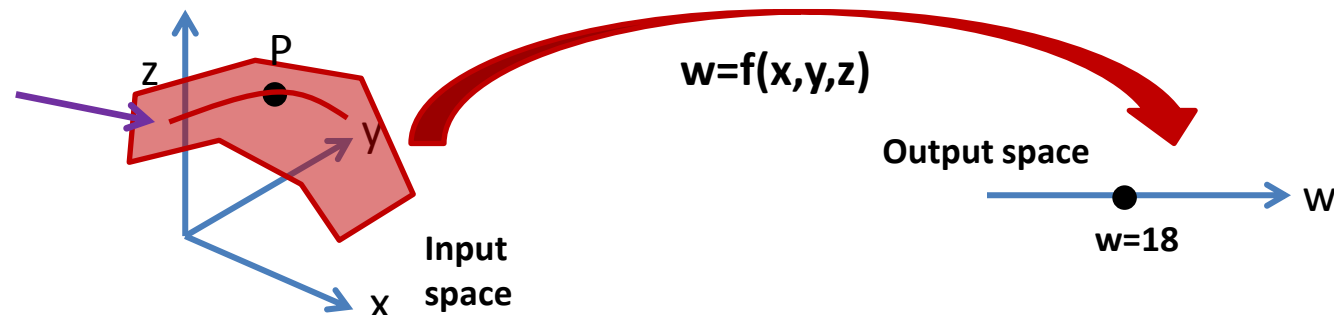
- (1) Consider a function of 3 variables: $w = f(x, y, z) \quad \mathbb{R}^3 \rightarrow \mathbb{R}^1$
- (2) Requires four dimensions to graph
- (3) Let's look at the level set $18 = f(x, y, z)$ which is the set of all input triples (x, y, z) that get sent to the output $w=18$
- (4) We can draw that **surface** in input space



- (5) Let's find a nice expression for the tangent plane to this level surface at input point $P=(x_0, y_0, z_0)$

Step 1: Imagine a curve $\vec{r}(t) = (x(t), y(t), z(t))$ on level surface: all inputs sent to output 18

So $f(x(t), y(t), z(t)) = 18$



Step 2: Let's differentiate both sides with respect to t

$$\frac{d}{dt} [f(x(t), y(t), z(t))] = \frac{d}{dt} [18] \quad \Rightarrow \quad \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} = 0$$

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) = 0 \quad \Rightarrow \quad \vec{\nabla} f \cdot \vec{r}'(t) = 0 \quad \text{At } t=t_0 \quad \vec{\nabla} f(t_0) \cdot \vec{r}'(t_0) = 0$$

Step 3: $\vec{\nabla} f(t_0) \cdot \vec{r}'(t_0) = 0$ says that the gradient is perpendicular to the tangent to every curve in the 18 level set passing through $P = (x_0, y_0, z_0)$

Step 4: Ah ha! So $\vec{\nabla} f(x_0, y_0, z_0)$ is normal to the tangent plane at the point $P = (x_0, y_0, z_0)$

Step 5: Remember that the tangent plane is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

So here is our snappy new formula for a plane tangent to the level surface at the point (x_0, y_0, z_0)

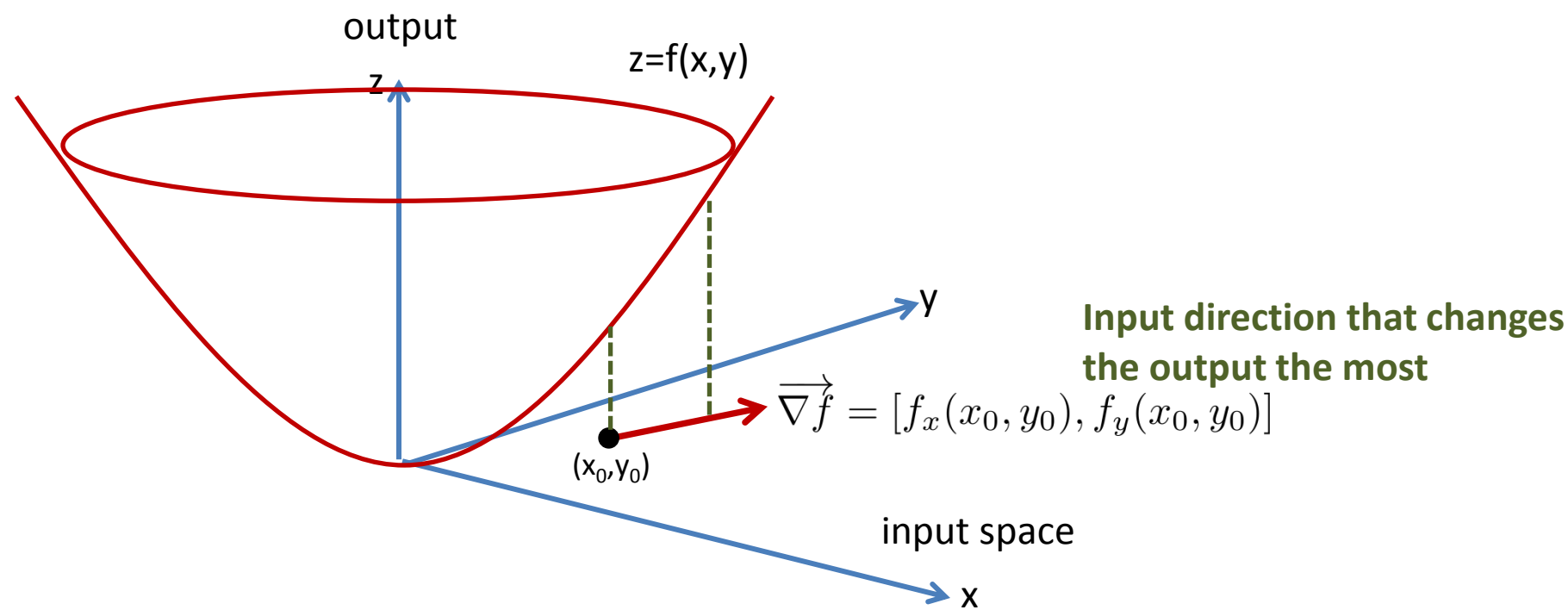
$$\vec{\nabla} f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

Section 14.6: Let's review what just happened: The gradient and tangent planes

So far, we have defined: $D_{\vec{u}}f(x, y) = \vec{\nabla} f \cdot \vec{u}$ [standing at input point (x, y) , this measures the change in f as we move in the direction $\vec{u}$]

And, we have found that f changes the most if we move in the same direction $\vec{u}$ as the gradient $\vec{\nabla} f$



Section 14.6: The gradient and tangent planes

Claim: for any function $f(x,y)$ or $f(x,y,z)$, [more generally, $f: \mathbb{R}^n \rightarrow \mathbb{R}^1$]
The gradient at input (x_0, y_0, z_0) is orthogonal to the level set passing through (x_0, y_0, z_0)

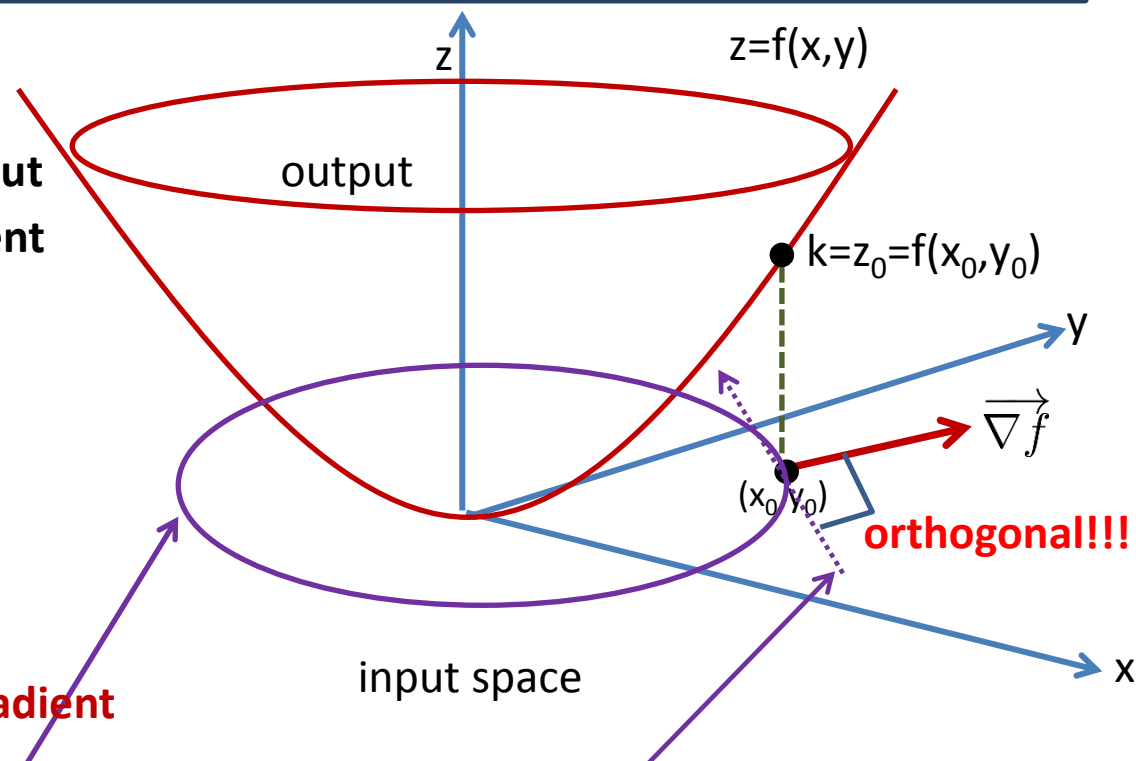
Wow! What does this even mean?

- (1) We have a function
- (2) At an input (x_0, y_0) we have an output
- (3) At an input (x_0, y_0) we have a gradient
- (4) At an input (x_0, y_0) we also have a level set corresponding to all inputs sent to the same output.
- (5) And we have a tangent to that level set at the input (x_0, y_0)

And I am claiming that tangent and gradient are orthogonal!!!

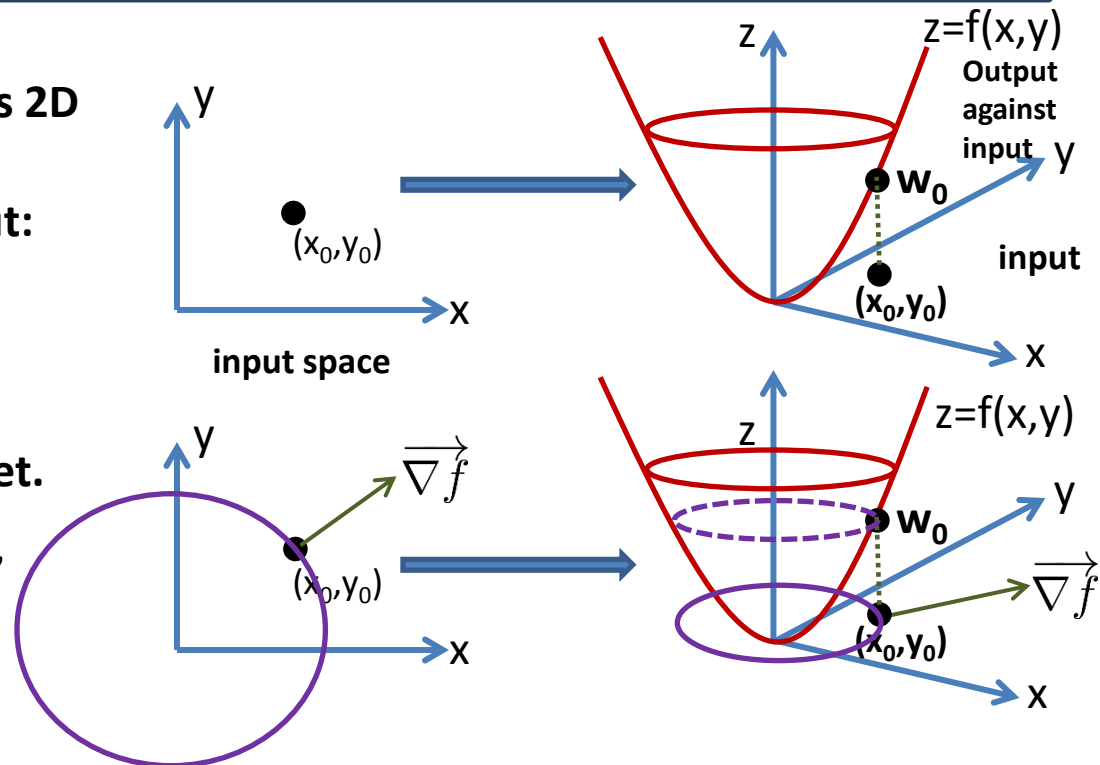
k level set = set of all input points sent to output $z=k$

Tangent to level curve at (x_0, y_0)



Claim: for any function $f(x,y)$ or $f(x,y,z)$, [more generally, $f: \mathbb{R}^n \rightarrow \mathbb{R}^1$]
The gradient at input (x_0, y_0, z_0) is orthogonal to the level set passing through (x_0, y_0, z_0)

- First, what does everything mean?
- Given a function $f(x,y)$, input space is 2D
- Given a point (x_0, y_0) in input space, we evaluate the function at the input: that is, let $w_0 = f(x_0, y_0)$
- What other points besides (x_0, y_0) in input space get sent to output $w_0 = f(x_0, y_0)$? This would be the w_0 level set.
- We could also ask, standing at (x_0, y_0) , **[IN INPUT SPACE!!!]** what direction increases the output the most?
This is the gradient $\vec{\nabla} f$



Here come the proof:

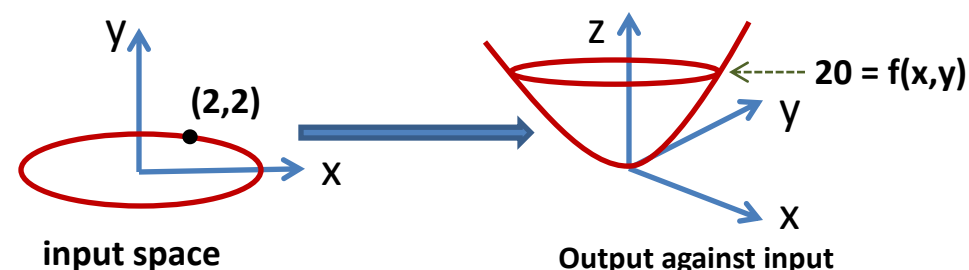
Step 1: Along the level curve, the output doesn't change, so $D_{\vec{u}} f(x_0, y_0) = 0$ when $\vec{u}$ is tangent to the level curve at (x_0, y_0)

Step 2: So $D_{\vec{u}} f(x_0, y_0) = \vec{\nabla} f \cdot \vec{u} = 0$ so $\vec{u}$ **must be orthogonal to** $\vec{\nabla} f$ **DONE!**

The gradient at input (x_0, y_0, z_0) is orthogonal to the level set passing through (x_0, y_0, z_0)

Example: Find the equation of the line tangent to the curve $x^2 + 4y^2 = 20$ at input point $(2,2)$

Solution: We attack this problem by turning the curve into the level set of a higher dimensional function. Once we do this, we will know that the tangent at $(2,2)$ is orthogonal to the gradient.



Step 1: Let $f(x,y) = x^2 + 4y^2$

Step 2: Then the “20” level set is the set of all input points sent to output “20”.

Step 3: So the gradient $\vec{\nabla} f = (2x, 8y) = (4, 16)$ is orthogonal to the tangent line at input $(2,2)$

Step 4: So the tangent line goes through $(2,2)$, and is orthogonal to the normal vector $(4,16)$

Step 5: But wait! We know the formula for a line going through (x_0, y_0) with normal $n=(a,b)$

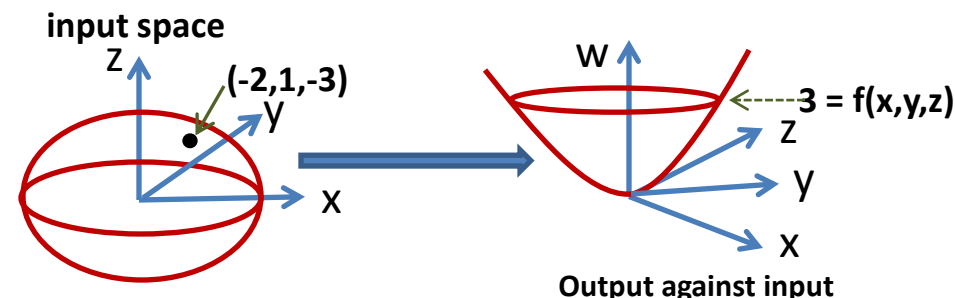
$$\vec{n} \cdot (x - x_0, y - y_0) = 0$$

$$\Rightarrow 4(x-2) + 16(y-2) = 0$$

The gradient at input (x_0, y_0, z_0) is orthogonal to the level set passing through (x_0, y_0, z_0)
 ...and this all works in 3D as well (and higher)

Find the equation of plane tangent to the ellipsoid $x^2/4 + y^2 + z^2/9 = 3$ at input point $(-2, 1, -3)$

Solution: We attack this problem by turning the ellipsoid into the level set of a higher dimensional function. Once we do this, we will know that the tangent plane at $(-2, 1, -3)$ is orthogonal to the gradient.



Step 1: Let $f(x, y, z) = x^2/4 + y^2 + z^2/9$

Step 2: Then the “3” level set is the set of all input points sent to output “3”.

Step 3: So the gradient $\vec{\nabla} f = ((1/2)x, 2y, 2z/9) \Big|_{(-2, 1, -3)} = (-1, 2, -2/3)$ is orthogonal to the tangent plane at input $(-2, 1, -3)$

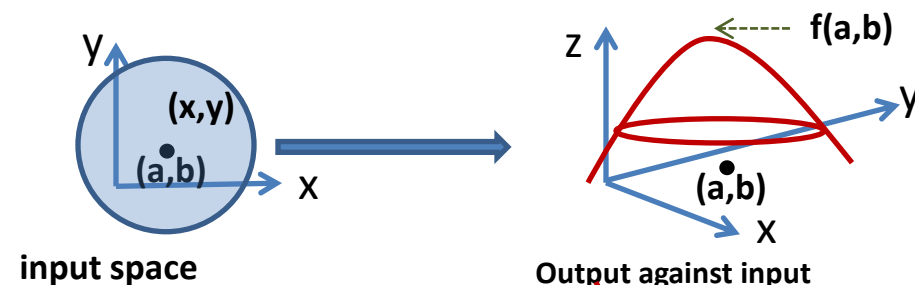
Step 4: So the tangent plane goes through $(-2, 1, -3)$, and is orthogonal to the normal vector $(-1, 2, -2/3)$

Step 5: We know the formula for a plane going through (x_0, y_0, z_0) with normal $n=(a, b, c)$

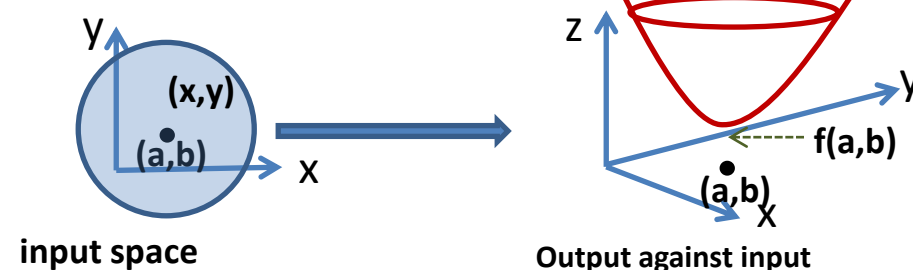
$$\vec{n} \cdot (x - x_0, y - y_0, z - z_0) = 0 \quad \longrightarrow \quad -1(x+2) + 2(y-1) - 2/3(z+3) = 0$$

Section 14.7: Maximum and Minimum values of Functions

Definition: A function of two variables has a **local maximum** at (a,b) if $f(a,b) \geq f(x,y)$ when (x,y) is near (a,b)

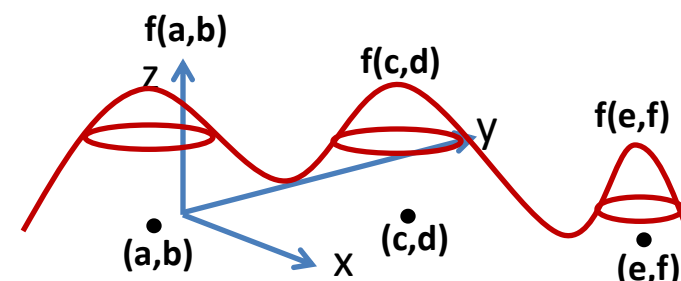


Definition: A function of two variables has a **local minimum** at (a,b) if $f(a,b) \leq f(x,y)$ when (x,y) is near (a,b)



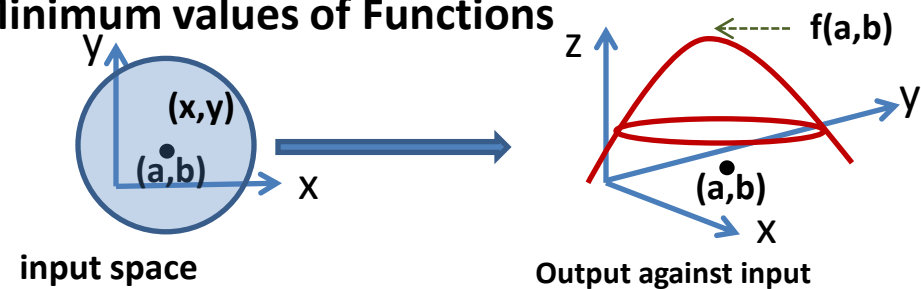
Definition: A local maximum at (a,b) is an **absolute maximum** if $f(a,b)$ is greater than or equal to f at every other point in input space

N.B.: There can be more than one absolute maximum
---if they have the same value
 $f(a,b) = f(c,d) > f(e,f)$



Section 14.7: Maximum and Minimum values of Functions

If $f(x,y)$ has a **local maximum** at (a,b) and continuous partial first derivatives, then $f_x(a,b)=f_y(a,b) = 0$



We prove that $f_x(a,b)=0$

Step 1: Let $g(x)=f(x,b)$.

Step 2: So g is a function of one variable, and has a maximum at $x=a$

Step 3: So that means that

$$\left. \frac{dg(x,b)}{dx} \right|_{x=a} = f_x(a,b) = 0 \quad \text{DONE!}$$

[Same proof for showing that $f_y(a,b) = 0$]

Definition: Given $f(x,y)$, a point (a,b) is a **critical point** if $f_x(a,b)=f_y(a,b) = 0$

Section 14.7: Maximum and Minimum values of Functions

Enough! Let's do some examples:

Example: Find the critical points of $f(x,y) = x^2 + y^2 - 2x - 6y + 14$

Solution: $f_x = 2x - 2$ so $f_x = 0$ when $x = 1$

$f_y = 2y - 6$ so $f_y = 0$ when $y = 3$



So there's one critical point at $(1,3)$

Example: Find the critical points of $f(x,y) = (1/3)x^3 - (3/2)x^2 + 2x + (1/3)y^3 - (5/2)y^2 + 6y + 85$

Solution: $f_x = x^2 - 3x + 2$ so $f_x = 0$ when $x = 1$ or $x = 2$

$f_y = y^2 - 5y + 6$ so $f_y = 0$ when $y = 2$ or $y = 3$



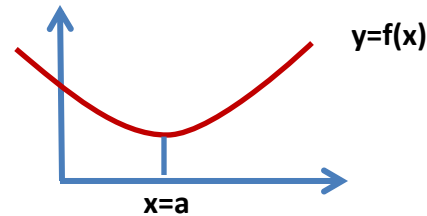
So there's four critical points $(1,2)$, $(2,2)$, $(1,3)$ and $(2,3)$

Fine. How can we tell if a critical point is a maximum or a minimum?

Section 14.7: Maximum and Minimum values of Functions

How can we tell if a critical point is a maximum or a minimum?

Good old-fashioned 1D:



$x=a$ is a critical point

$$f'_x(x=a) = 0$$

$$f''_{xx}(x=a) > 0$$

Unfortunately, it's not
as simple in 2D:

The painful example: A saddle surface.

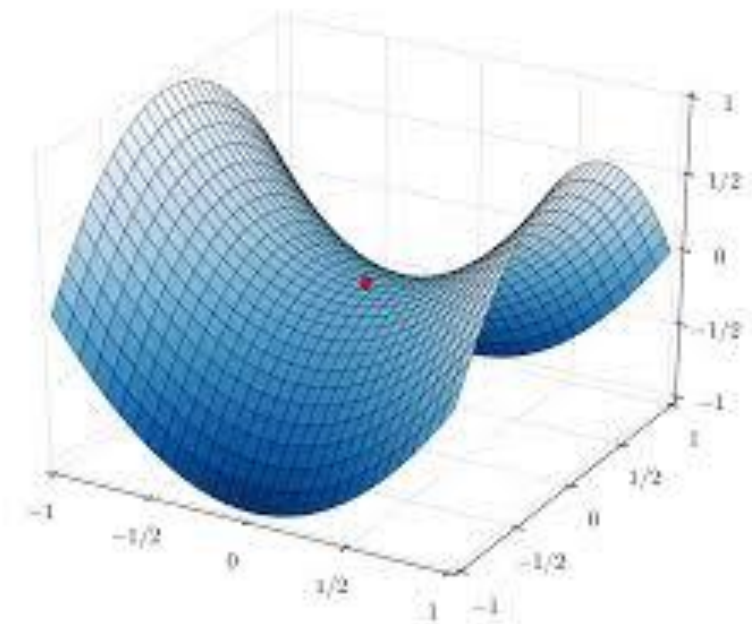
At the red critical point,
 $f'_x(\text{red_point}) = f'_y(\text{red_point}) = 0$

But

$$f''_{xx}(\text{red_point}) > 0 \text{ and } f''_{yy}(\text{red_point}) < 0$$

And this point is neither a minimum nor maximum!

We need a different test to determine if maximum or minimum



Section 14.7: Maximum and Minimum values of Functions

The “D” test to determine if maximum or minimum

Suppose the 2nd partial derivatives of $f(x,y)$ are continuous on a disk around (a,b)

Suppose $f_x(a,b) = f_y(a,b) = 0$ (in other words, (a,b) is a critical point)

Let $D = D(a,b) = f_{xx}(a,b) * f_{yy}(a,b) - [f_{xy}(a,b)]^2$ **[Don't worry—no one can remember this!]**

Then

- (a) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then (a,b) is a local minimum
- (b) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then (a,b) is a local maximum
- (c) If $D(a,b) < 0$ then it is neither a local minimum nor a local maximum