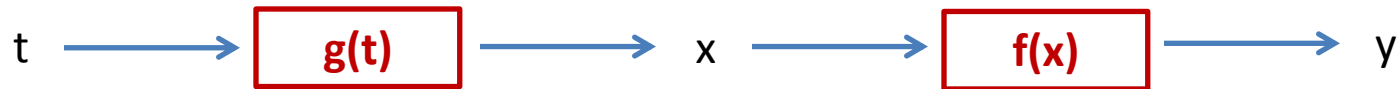


Section 14.5: The Chain Rule (Review)

**This is beautiful—
but it requires some
conceptual thinking.
Let's go!**

Begin by recalling the 1D chain rule:



Then if we ask “how does the output y change as we change the input t ”,

Then we are asking for $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$

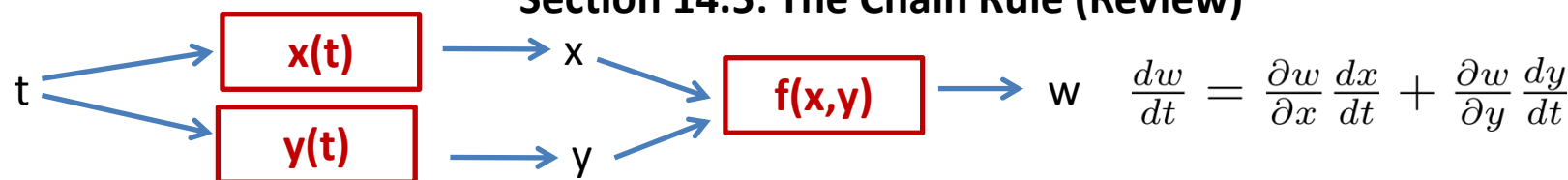
Note the similarity between this and our equation for the Slope of a tangent line

$$f(x) = f(a) + \left. \frac{df}{dx} \right|_a (x - a) \leftarrow \text{Equation for Line tangent to } (a, f(a))$$

$$f(x) - f(a) = \left. \frac{df}{dx} \right|_a (x - a)$$

Which says that the change in output $(f(x)-f(a))$ $\longrightarrow \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$
 equals
 the derivative times the change in input $(x-a)$ $\nearrow$

Section 14.5: The Chain Rule (Review)



2d: Formula for the tangent plane

$$f(x, y) = f(a, b) + \left. \frac{\partial f}{\partial x} \right|_{a,b} (x-a) + \left. \frac{\partial f}{\partial y} \right|_{a,b} (y-b) \leftarrow \text{Equation for plane tangent to } (a, b, f(a, b))$$

$$f(x, y) - f(a, b) = \left. \frac{\partial f}{\partial x} \right|_{a,b} (x - a) + \left. \frac{\partial f}{\partial y} \right|_{a,b} (y - b)$$

Which says that the change in output ($f(x,y)-f(a,b)$)

equals

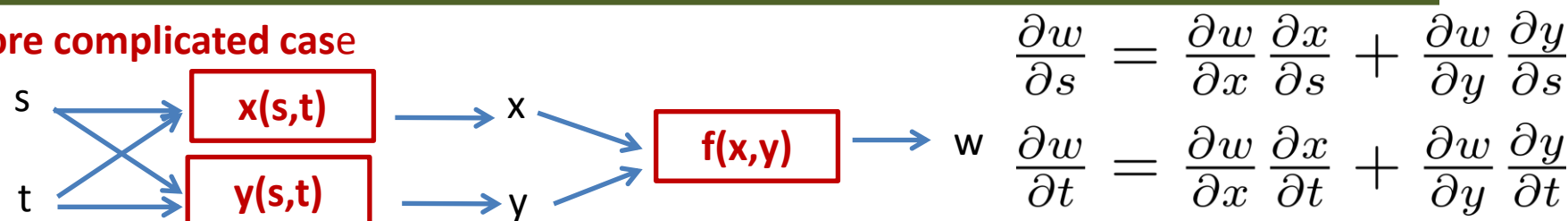
the partial x derivative times the change in x input ($x-a$)

+

the partial y derivative times the change in y input ($y-b$)

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

A more complicated case

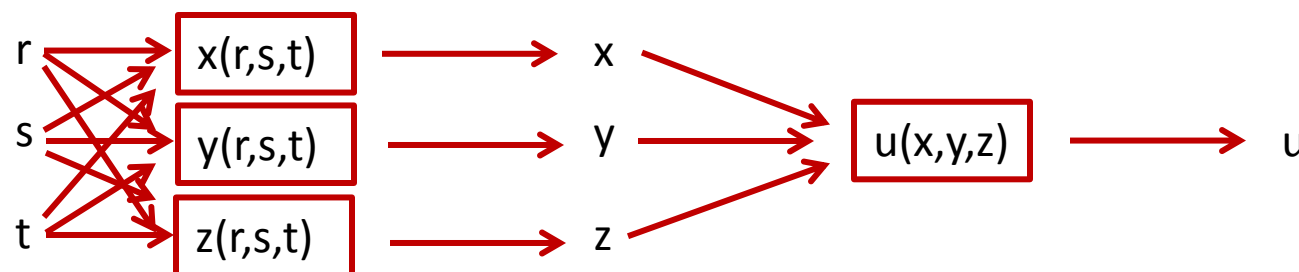


Section 14.5:

Example: Let $u = x^4y + y^2z^3$ where $x=rse^t$, $y = r s^2 e^{-t}$ and $z = r^2 s \sin t$

Find $\frac{\partial u}{\partial t}$

Answer: Step 1:
Make a
drawing!!!!



**Step 2: Write the
chain of all paths**

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t}$$

**Step 3: Evaluate
all the partials**

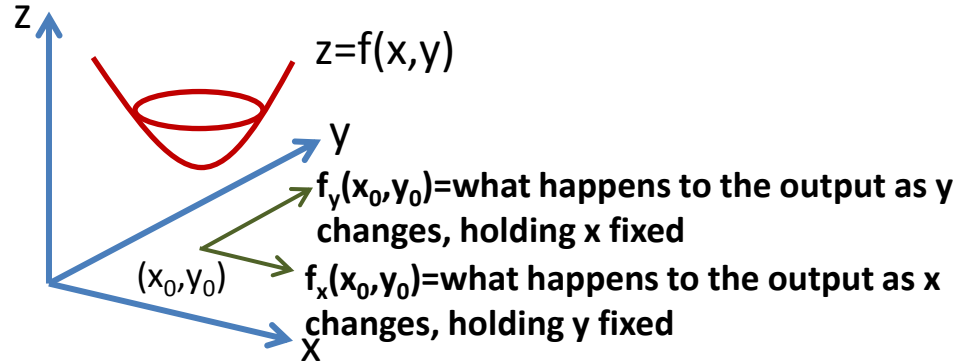
$$= (4x^3y)(rse^t) + (x^4 + 2yz^3)(-rs^2e^{-t}) + (y^23z^2)(r^2s \cos t)$$

Step 4: Substitute to get rid of all the intermediate variables

$$= (4(rse^t)^3(rs^2e^{-t}))(rse^t) + ((rse^t)^4 + 2(rs^2e^{-t})(r^2s \sin t)^3)(-rs^2e^{-t}) + ((rs^2e^{-t})^2 3(r^2s \sin t)^2)(r^2s \cos t)$$

Section 14.6: Directional Derivatives and the Gradient!

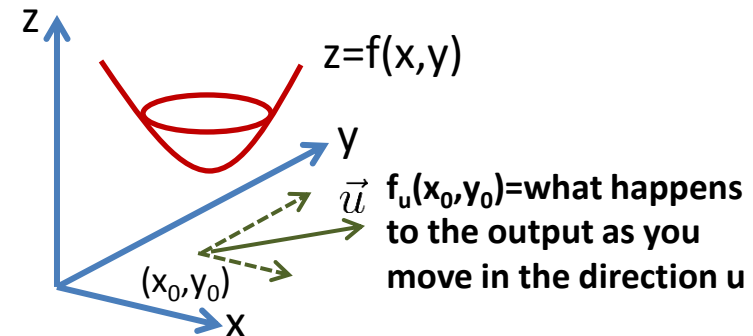
We now know what partial derivatives f_x and f_y mean



But what if you go in a different direction $\vec{u}$?
You could ask “how does the output

change as you move in the direction u ?

This is called the “**directional derivative**”



Definition:
$$D_{\vec{u}}f(x_0, y_0) \equiv \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

where $\vec{u}$ is a unit vector with components $\vec{u} = (a, b) \quad |\vec{u}| = 1$

Looks a lot like:
$$f_x(a, b) \equiv \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h} \quad f_y(a, b) \equiv \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}$$

$$\vec{u} = \vec{i} = (a, 0) = (1, 0) \quad \vec{u} = \vec{j} = (0, b) = (0, 1)$$

Section 14.6: Directional Derivatives and the Gradient!

Is there an easy way to calculate:

$$D_{\vec{u}} f(x_0, y_0) \equiv \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

where $\vec{u}$ is a unit vector with components $\vec{u} = (a, b)$ $|\vec{u}| = 1$

Yes! And it's so cool!

Claim: $D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$!!!

In other words: to take the directional derivative of f in the direction $\vec{u}$, then

$$D_{\vec{u}} f(x_0, y_0) = (f_x, f_y) \cdot \vec{u}$$

Question: why does $\vec{u}$ need to be a unit vector?

Let's use it before proving it: Suppose $f(x, y) = x^3 + 3y^4$. Find $D_{\vec{u}} f$ if $\vec{u} = (3/5, 4/5)$ at input $(2, 3)$

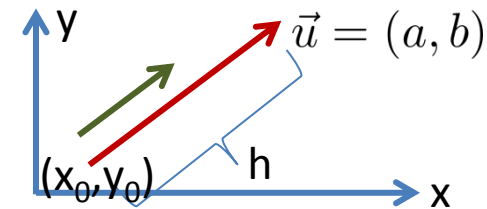
Solution: $D_{\vec{u}} f = (f_x, f_y) \cdot (\frac{3}{5}, \frac{4}{5}) = (3x^2, 12y^3) \cdot (\frac{3}{5}, \frac{4}{5}) = 3\frac{3}{5}(x^2) + 12\frac{4}{5}(y^3)$

$$D_{\vec{u}} f(2, 3) = (f_x(2, 3), f_y(2, 3)) \cdot (\frac{3}{5}, \frac{4}{5}) = 3\frac{3}{5}(2^2) + 12\frac{4}{5}(3^3)$$

Let's prove it $D_{\vec{u}}f(x_0, y_0) = (f_x, f_y) \cdot \vec{u}$

Step 1: Let's parameterize by t the green vector leaving (x_0, y_0) and headed in the direction $\vec{u}$

$$\longrightarrow (x(t), y(t)) = (x_0 + ta, y_0 + tb)$$



Step 2: We have value for the function $f(x, y)$ at every point on the green vector $f(x(t), y(t)) = f(x_0 + ta, y_0 + tb)$ * * *

Our strategy: Evaluate the derivative of both sides of * * * at the input $t=0$

Step 3: The expression on the right-hand-side of * * * is a function of only one variable (namely, t)---since (a, b) and (x_0, y_0) are all fixed, so we can define a new function

$$g(t) = f(x_0 + ta, y_0 + tb)$$

Step 4: So the derivative of the right-hand-side of $g(t)$ at $t=0$ is given by

$$g'(0) = \lim_{t \rightarrow 0} \frac{g(t) - g(0)}{t} = \lim_{t \rightarrow 0} \frac{f(x_0 + ta, y_0 + tb) - f(x_0, y_0)}{t} = D_u f(x_0, y_0)$$

The same!

Step 5: On the other hand, we can rewrite the left-hand-side of * * * as $g(t) = f(x(t), y(t))$

Step 6: Time for the chain rule: $\frac{dg}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = f_x a + f_y b$

So the derivative of the left-hand-side is

$$\left. \frac{dg}{dt} \right|_{t=0} = f_x \bigg|_{t=0} a + f_y \bigg|_{t=0} b = f_x(x_0, y_0)a + f_y(x_0, y_0)b = (f_x, f_y) \cdot \vec{u}$$

We now introduce a
snappy new notation:

Definition: if f is a function of two variables $f(x,y)$, then we
define the gradient $\vec{\nabla} f \equiv (f_x, f_y)$ $\leftarrow$ **Note:** this is a **vector**

Example: $f(x,y) = \sin(xy)$ Find $\vec{\nabla} f$

Solution: $\vec{\nabla} f = (f_x, f_y) = (y \cos(xy), x \cos(xy))$

So now we have a new, nifty notation: $D_u f(x, y) = \vec{\nabla} f \cdot \vec{u}$

Higher dimensions:

Obviously, we can keep this going to higher dimensions:

if $f(x,y,z,w) = x^3 y^2 + xyzw + w^3$ find $\vec{\nabla} f$ and find $D_u f(x, y, z, w) = \vec{\nabla} f \cdot \vec{u}$ if $u=(1,1,1,2)/\sqrt{7}$

Solution: $\vec{\nabla} f = (f_x, f_y, f_z, f_w) = (3x^2 y^2 + yzw, 2x^3 y + xzw, xyw, xyz + 3w^2)$

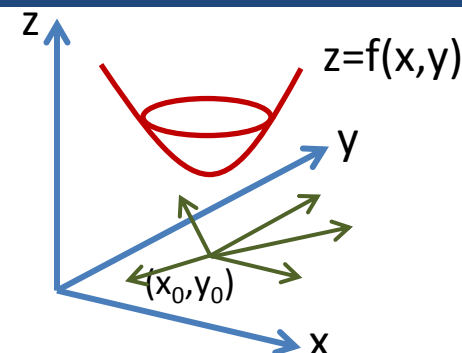
$$D_u f = \vec{\nabla} f \cdot \vec{u} = [(3x^2 y^2 + yzw, 2x^3 y + xzw, xyw, xyz + 3w^2)] \cdot \left(\frac{1, 1, 1, 2}{\sqrt{7}} \right)$$

And now comes an amazing moment:

$$D_u f(x, y) = \vec{\nabla} f \cdot \vec{u}$$

Gives the directional derivative in the direction u

Standing at (x_0, y_0) in which direction should you go to change f the most?

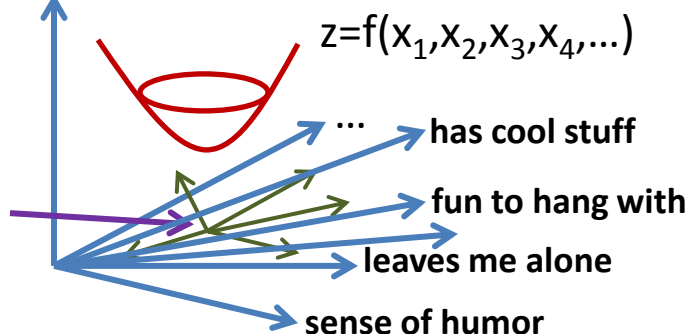


Example: f = how much I like my roommate/brother/sister
 $= f(\text{sense of humor, leaves me alone, fun to hang with, got cool stuff, ...})$

If you could pick any unit vector u , what direction would you go to maximize how much you like your roommate/brother/sister?

How much you like them

What you have now



The mathematical version of this question

What direction makes $D_u f(x, y) = \vec{\nabla} f \cdot \vec{u}$ the biggest?

Answer: well,

$$D_u f(x, y) = \vec{\nabla} f \cdot \vec{u} = |\vec{\nabla} f| |\vec{u}| \cos \theta$$

Which is biggest when $\cos \theta$ is biggest
 which happens when $\vec{\nabla} f$ and $\vec{u}$

point in the same direction!

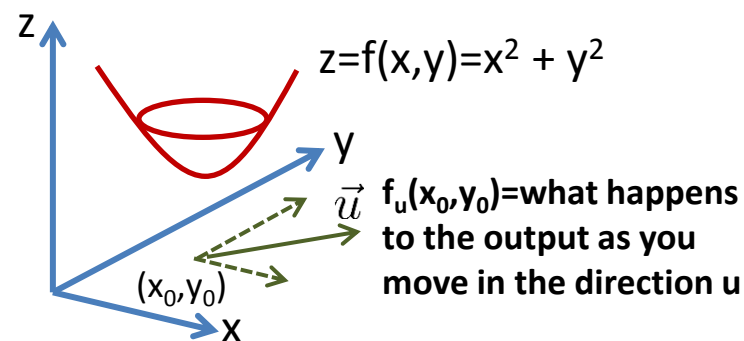
Again: $D_{\vec{u}}f(x, y) = \vec{\nabla}f \cdot \vec{u}$ **is biggest when $\vec{\nabla}f$ and $\vec{u}$ point in the same direction!**

Again: standing at (x_0, y_0) , you make the biggest change in the output when you move in the direction $\vec{\nabla}f$ in input space!

Example: suppose $f(x, y) = x^2 + y^2$

Standing at $(1, 1)$ the output is $f(1, 1) = 2$

What direction should you move to increase f the most?



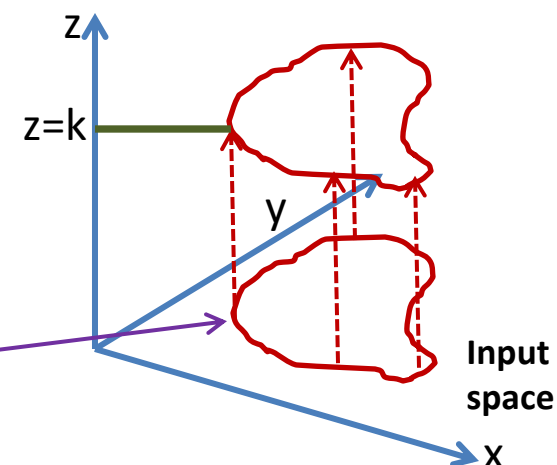
Solution: $\vec{\nabla}f(x, y) = (f_x, f_y) = (2x, 2y)$ at input $(1, 1)$ this gives the direction $(2, 2)$

$$D_u f(x, y) = \vec{\nabla} f \cdot \vec{u}$$

This question helps figure out if you get what is going on

Consider the function $f(x, y)$. This is a mapping from $\mathbb{R}^2 \rightarrow \mathbb{R}^1$

Now, consider the level set $k=f(x, y)$, which is the set of all points in input space that get sent to the output k .



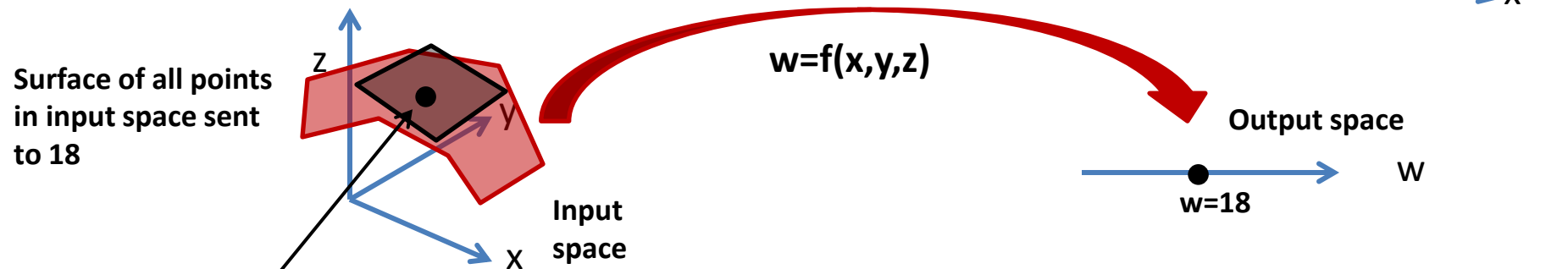
Question: what is the value of the directional derivative $D_u f(x, y)$ as you move along the k level set?

Answer: Zero!!! Because the output does not change as you move on a level set!!!

Let's use these new ideas to think about tangent planes again

$$D_u f(x, y) = \vec{\nabla} f \cdot \vec{u}$$

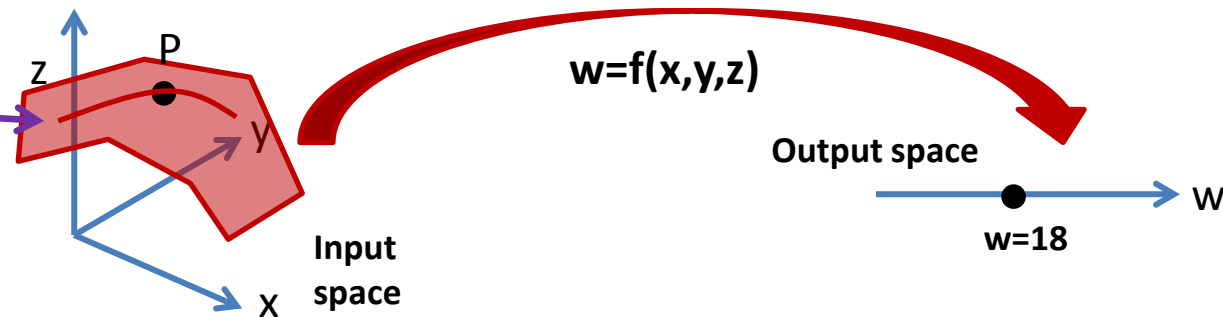
- (1) Consider a function of 3 variables: $w = f(x, y, z) \quad \mathbb{R}^3 \rightarrow \mathbb{R}^1$
- (2) Requires four dimensions to graph
- (3) Let's look at the level set $18 = f(x, y, z)$ which is the set of all input triples (x, y, z) that get sent to the output $w=18$
- (4) We can draw that **surface** in input space



- (5) Let's find a nice expression for the tangent plane to this level surface at input point $P = (x_0, y_0, z_0)$

Step 1: Imagine a curve $(x(t), y(t), z(t))$ on level surface: all inputs sent to output 18

So $f(x(t), y(t), z(t)) = 18$



Step 2: Let's differentiate both sides with respect to t

$$\frac{d}{dt} [f(x(t), y(t), z(t))] = \frac{d}{dt} [18] \quad \Rightarrow \quad \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} = 0$$

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) = 0 \quad \Rightarrow \quad \vec{\nabla} f \cdot \vec{r}'(t) = 0 \quad \text{At } t=t_0 \quad \vec{\nabla} f(t_0) \cdot \vec{r}'(t_0) = 0$$

Step 3: $\vec{\nabla} f(t_0) \cdot \vec{r}'(t_0) = 0$ says that the gradient is perpendicular to every curve in the 18 level set passing through $P = (x_0, y_0, z_0)$

Step 4: Ah ha! So $\vec{\nabla} f(x_0, y_0, z_0)$ is normal to the tangent plane at the point $P = (x_0, y_0, z_0)$

Step 5: Remember that the tangent plane is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

So here is our snappy new formula for a plane tangent to the level surface at the point (x_0, y_0, z_0)

$$\vec{\nabla} f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$