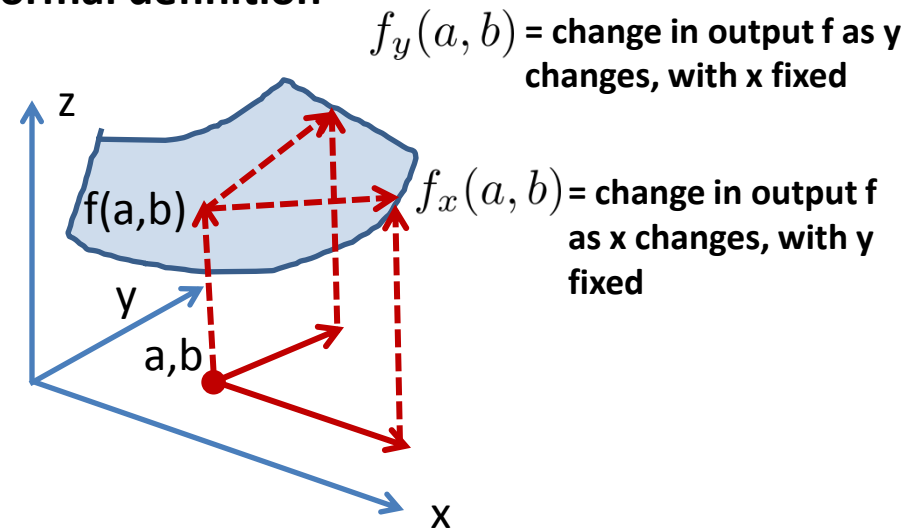


Section 14.3: The formal definition

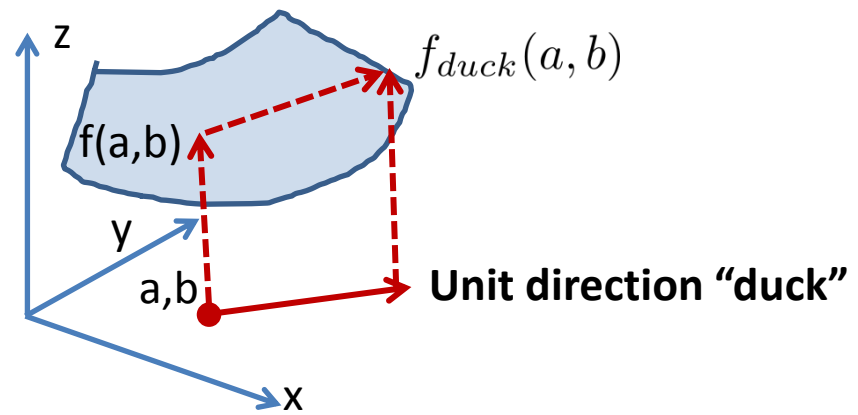
Let's review partial derivatives:

$$f_x(a, b) \equiv \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$$

$$f_y(a, b) \equiv \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}$$



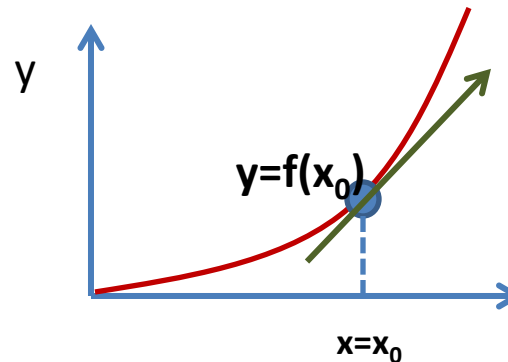
So, you should think of the partial derivative in the “duck” direction as how much the function changes as you move from (a,b) in the unit vector direction “duck”



Section 14.4: Tangent Planes and linear approximations

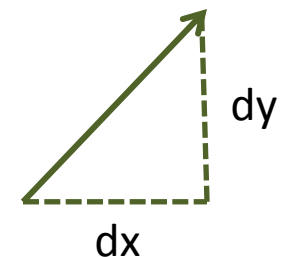
Let's recall 1D Calculus: $y=f(x)$

Tangent line at $(x_0, f(x_0))$ touches the graph $y=f(x)$ at only one point in near $(x_0, f(x_0))$



Tangent line with slope $f'(x_0)$ going through the point $(x_0, f(x_0))$

The tangent line is given by $y - f(x_0) = slope(x - x_0) = \left. \frac{df}{dx} \right|_{x_0} (x - x_0)$

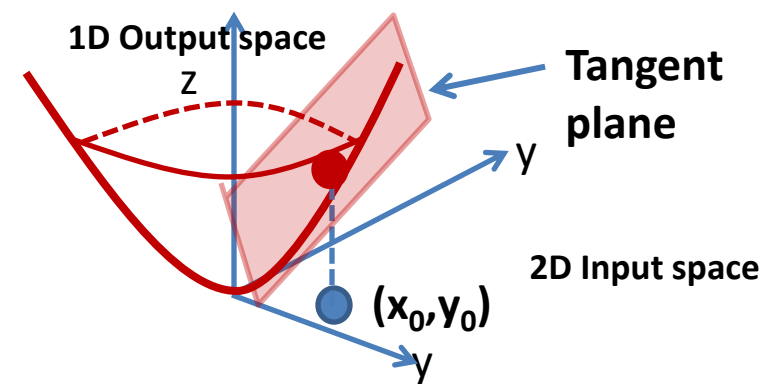


Tangent vector = $(dx, dy) = (1, dy/dx) = (1, f'(a))$

We want to construct a similar idea for functions of two (or more variables):

The Tangent Plane

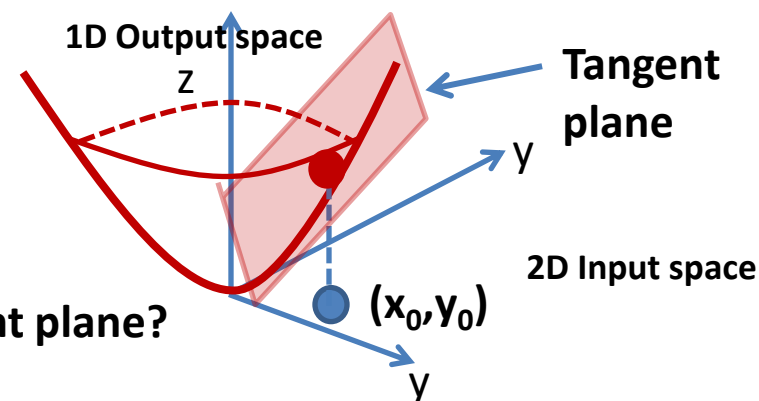
Tangent plane at $(x_0, y_0, f(x_0, y_0))$ touches the graph $z=f(x, y)$ at only one point is near $(x_0, y_0, f(x_0, y_0))$



Section 14.4: Tangent Planes and linear approximations

Tangent plane at $(x_0, y_0, f(x_0, y_0))$ touches the graph $z=f(x,y)$ at only one point is near $(x_0, y_0, f(x_0, y_0))$

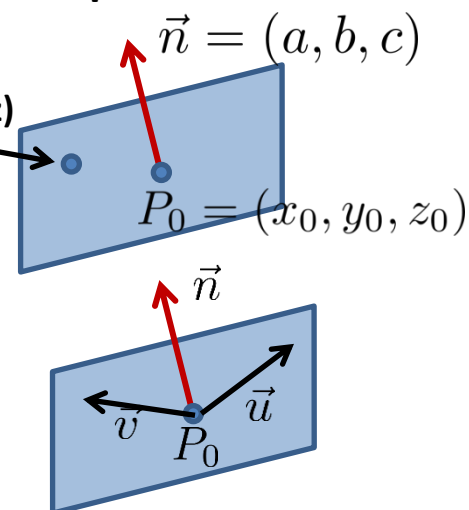
How are we going to find an equation for the tangent plane?



Idea #1! Do you remember we had a formula for a plane going through the point (x_0, y_0, z_0) with normal vector (a, b, c) ?

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Arbitrary point (x, y, z) on plane



Realization 1: we know the point P_0 where the tangent plane touches the surface: $P_0 = (x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$

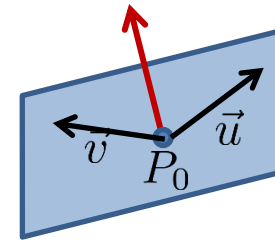
Realization 2: if we had two vectors $\vec{u}$ and $\vec{v}$ in the tangent plane, we could take their cross product to find the normal $\vec{n} = \vec{u} \times \vec{v}$

Section 14.4: Tangent Planes and linear approximations

Formula for a plane going through the point

(x_0, y_0, z_0) with normal vector (a, b, c)

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$



Tangent point is $P_0 = (x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$

How can we find two vectors in the tangent plane?

Idea #2: We can slice the graph of $z=f(x,y)$ with a plane $y=y_0$

This gives a purple curve whose y coordinate never changes and lies on the surface.

Tangent vector with slope $\left. \frac{\partial f}{\partial x} \right|_{x_0, y_0}$

So this purple curve is the graph of $(x, y_0, f(x, y_0))$

And the x partial derivative of this purple curve at (x_0, y_0) gives the slope $\left. \frac{df}{dx} \right|_{x_0, y_0}$ of the tangent vector

at P_0 lying in the slicing plane

So one tangent vector is

$$\vec{v} = \left(1, 0, \left. \frac{\partial f}{\partial x} \right|_{x_0, y_0} \right)$$

Section 14.4: Tangent Planes and linear approximations

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Tangent point is $P_0 = (x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$

Obtain another tangent vector by slicing the graph of $z=f(x,y)$ with a plane $x=x_0$

This gives a purple curve whose x coordinate never changes and lies on the surface.

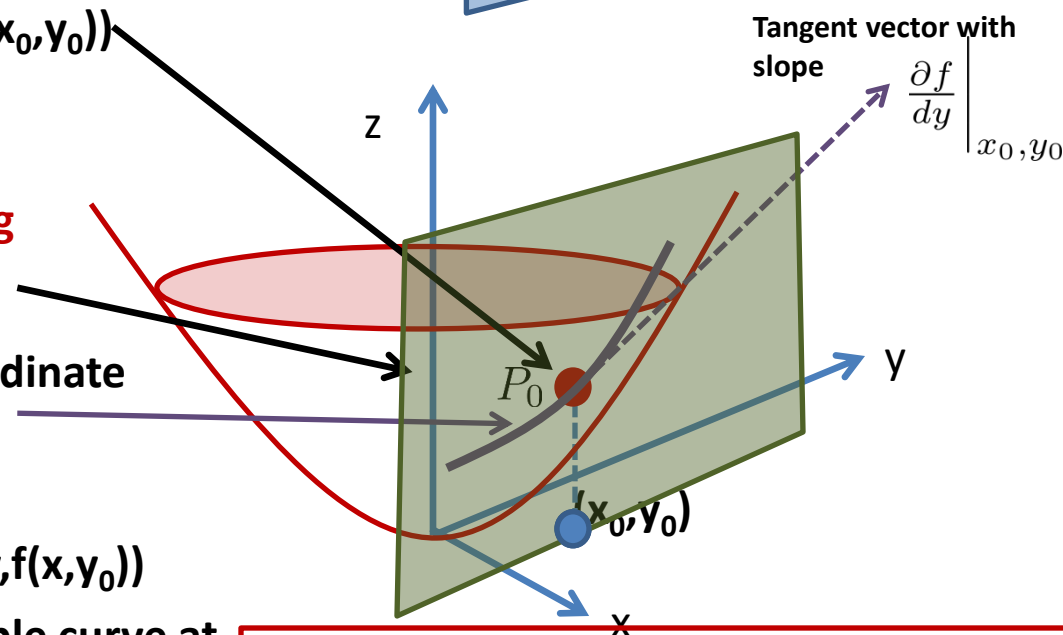
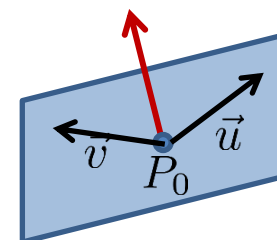
So this purple curve is the graph of $(x_0, y, f(x_0, y))$

And the y partial derivative of this purple curve at (x_0, y_0) gives the slope $\left. \frac{df}{dy} \right|_{x_0, y_0}$ of the tangent vector

at P_0 lying in the slicing plane

So one tangent vector is

$$\vec{v} = \left(0, 1, \left. \frac{\partial f}{\partial y} \right|_{x_0, y_0} \right)$$



Section 14.4: Tangent Planes and linear approximations

One tangent vector is $\vec{u} = (1, 0, \left. \frac{\partial f}{\partial x} \right|_{x_0, y_0})$

One tangent vector is $\vec{v} = (0, 1, \left. \frac{\partial f}{\partial y} \right|_{x_0, y_0})$

$$\vec{n} = (a, b, c) = \vec{u} \times \vec{v} = (1, 0, \left. \frac{\partial f}{\partial x} \right|_{x_0, y_0}) \times (0, 1, \left. \frac{\partial f}{\partial y} \right|_{x_0, y_0}) = (-\left. \frac{\partial f}{\partial x} \right|_{x_0, y_0}, -\left. \frac{\partial f}{\partial y} \right|_{x_0, y_0}, 1) \leftarrow \text{check this!}$$

Back to our formula for a plane:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

(I temporarily
stopped writing $\left. \right|_{x_0, y_0}$)

Substitute everybody in:

$$-\left. \frac{\partial f}{\partial x} \right|_{x_0, y_0} (x - x_0) + -\left. \frac{\partial f}{\partial y} \right|_{x_0, y_0} (y - y_0) + 1(z - f(x_0, y_0)) = 0$$

Solve for z (remembering that $z_0 = f(x_0, y_0)$):

$$z - f(x_0, y_0) = \left. \frac{\partial f}{\partial x} \right|_{x_0, y_0} (x - x_0) + \left. \frac{\partial f}{\partial y} \right|_{x_0, y_0} (y - y_0)$$

**Formula for the
tangent plane**

Section 14.4: Tangent Planes and linear approximations

**Formula for the
tangent plane**

$$z - f(x_0, y_0) = \left. \frac{\partial f}{\partial x} \right|_{x_0, y_0} (x - x_0) + \left. \frac{\partial f}{\partial y} \right|_{x_0, y_0} (y - y_0)$$

**Notice how much it looks like
our formula for the slope of a
tangent line:**

$$y - f(x_0) = \text{slope}(x - x_0) = \left. \frac{df}{dx} \right|_{x_0} (x - x_0)$$

Example: Find the equation for the plane tangent to $z = 2x^2 + y^2$ at the input point $x_0=1$ $y_0=1$

Solution: Step 1: find the point on the surface at the input $x_0=1$ $y_0=1$

$$z_0 = f(x_0, y_0) = f(1, 1) = 2(1)^2 + 1^2 = 3$$

Step 2: find the partial derivatives at the input point:

$$f_x = 4x \text{ so at input } x_0=1 \ y_0=1 \ f_x=4$$

$$f_y = 2y \text{ so at input } x_0=1 \ y_0=1 \ f_y=2$$

Step 3: put them into your equation for the tangent plane: $z-3 = 4(x-1) + 2(y-1)$

Section 14.4: Tangent Planes and linear approximations

Why stop there?

1d: Slope of a tangent line

$$f(x) = f(a) + \left. \frac{df}{dx} \right|_a (x - a) \leftarrow \text{Equation for Line tangent to } f(a)$$

2d: Formula for the tangent plane

$$f(x, y) = f(a, b) + \left. \frac{\partial f}{\partial x} \right|_{a,b} (x - a) + \left. \frac{\partial f}{\partial y} \right|_{a,b} (y - b) \leftarrow \text{Equation for plane tangent to } f(a, b)$$

3d: Formula for the tangent “hyperplane”

$$f(x, y, z) = f(a, b, c) + \left. \frac{\partial f}{\partial x} \right|_{a,b,c} (x - a) + \left. \frac{\partial f}{\partial y} \right|_{a,b,c} (y - b) + \left. \frac{\partial f}{\partial z} \right|_{a,b,c} (z - c)$$

Equation for hyperplane tangent to $f(a, b, c)$

“n”d: Formula for the tangent “hyperplane”

$$f(x_1, \dots, x_n) = f(a_1, a_2, \dots, a_n) + \left. \frac{\partial f}{\partial x_1} \right|_{a_1, a_2, \dots, a_n} (x_1 - a_1) + \left. \frac{\partial f}{\partial x_2} \right|_{a_1, a_2, \dots, a_n} (x_2 - a_2) + \dots + \left. \frac{\partial f}{\partial x_n} \right|_{a_1, a_2, \dots, a_n} (x_n - a_n)$$

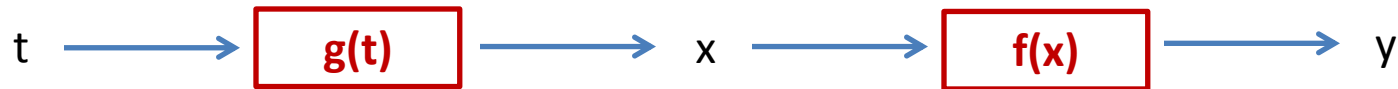
Equation for hyperplane tangent to $f(a_1, a_2, \dots, a_n)$

(No—I can’t draw it!)

Section 14.5: The Chain Rule

**This is beautiful—
but it requires some
conceptual thinking.
Let's go!**

Begin by recalling the 1D chain rule:

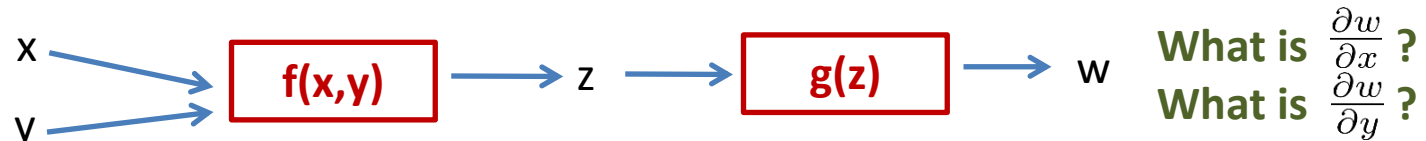


Then if we ask “how does the output y change as we change the input t ”,

Then we are asking for $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$

Example 1:

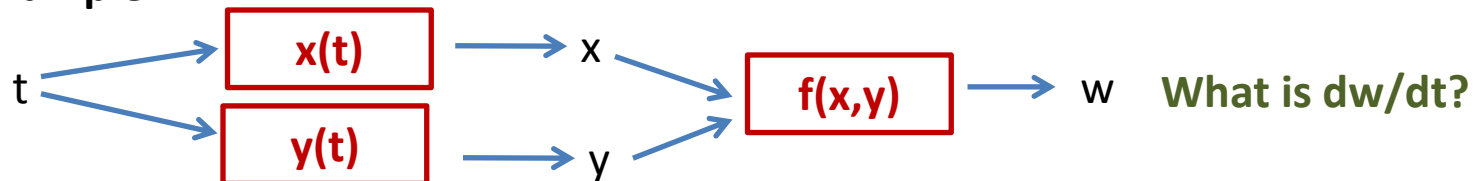
What does mean
when more than
one variable is
involved as input?



What is $\frac{\partial w}{\partial x}$?
What is $\frac{\partial w}{\partial y}$?

Example 2:

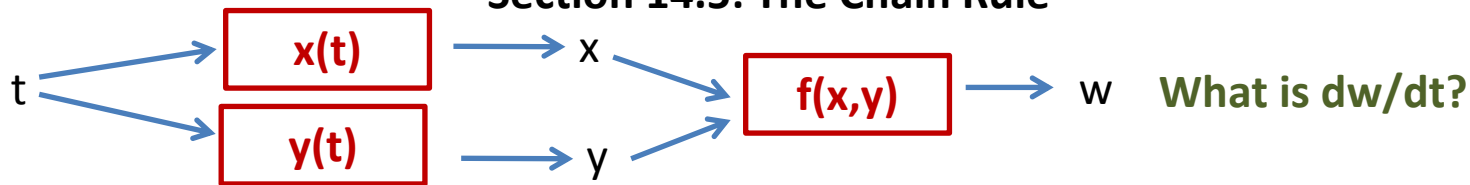
**Many different
diagrams to draw!**



What is dw/dt ?

The book has many examples of diagrams. They are **all** the same thing!

Section 14.5: The Chain Rule

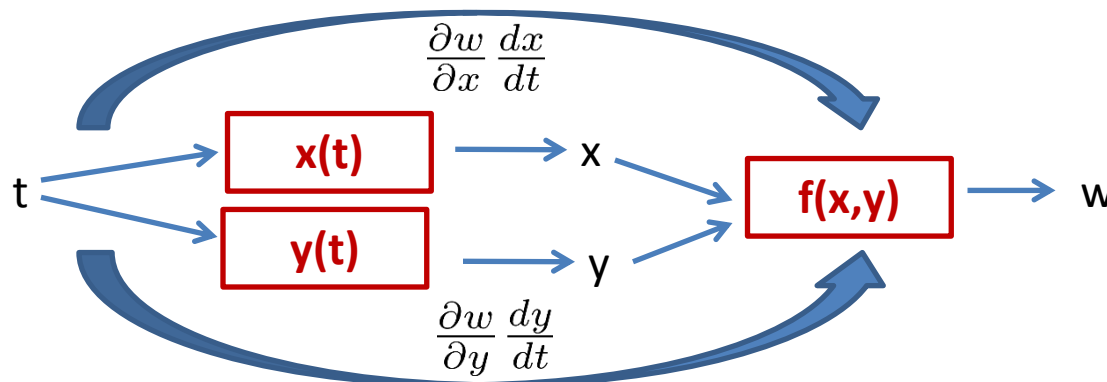


Claim: $\frac{dw}{dt}$ = Change in output w as input t changes $= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$

First question: Why do I write “d” on the left but partial signs ∂ on the right?

Answer: because w depends on only one variable “ t ”, hence it’s a simple derivative

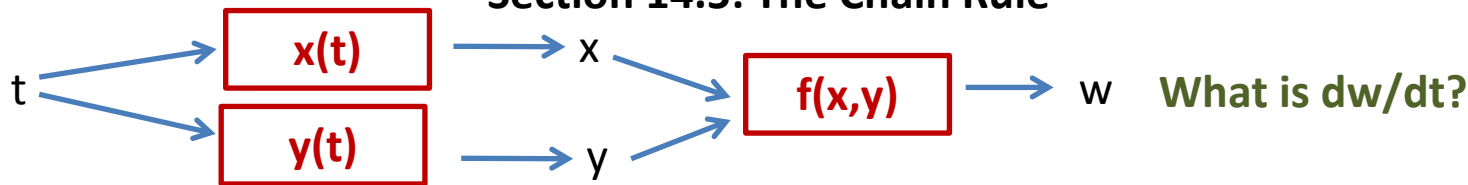
Second question: What is the meaning of the two terms that are added?:



Answer: they represent contributions from two different paths through the diagram

Third question: Why do we **add** the contributions from the two paths? First, an example

Section 14.5: The Chain Rule



Claim: $\frac{dw}{dt}$ = Change in output w as input t changes = $\frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$

**Before proving this,
let's do an example**

If $w(x,y)=x^3y + 3xy^4$ and $x=\sin(2t)$ and $y=\cos(t)$, find $\frac{dw}{dt}$

Solution:

$$\begin{aligned}
 \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\
 &= (3x^2y + 3y^4)(2 \cos 2t) + (x^3 + 12xy^3)(-\sin t) \\
 &= [3(\sin 2t)^2(\cos t) + 3(\cos t)^4] (2 \cos 2t) \\
 &\quad + [(\sin 2t)^3 + 12(\sin 2t)(\cos t)^3] (-\sin t)
 \end{aligned}$$

Section 14.5: The Chain Rule

Again: $\frac{dw}{dt} = [3(\sin 2t)^2(\cos t) + 3(\cos t)^4] (2 \cos 2t) + [(\sin 2t)^3 + 12(\sin 2t)(\cos t)^3] (-\sin t)$

Note—you could do this the “old way”

Solution (old way)-- If $w(x,y)=x^3y + 3xy^4$ and $x=\sin(2t)$ and $y=\cos(t)$, find $\frac{dw}{dt}$
substitute: $w(x, y) = (\sin 2t)^3 \cos t + 3 \sin 2t (\cos t)^4$

$$\frac{dw}{dt} = [3(\sin 2t)^2 (2 \cos 2t)] \cos t + (\sin(2t))^3 (-\sin t)$$

$$3(2 \cos 2t)(\cos t)^4 + 3 \sin 2t [4(\cos t)^3] (-\sin t)$$


They are the same! And the new way is much easier in complicated cases

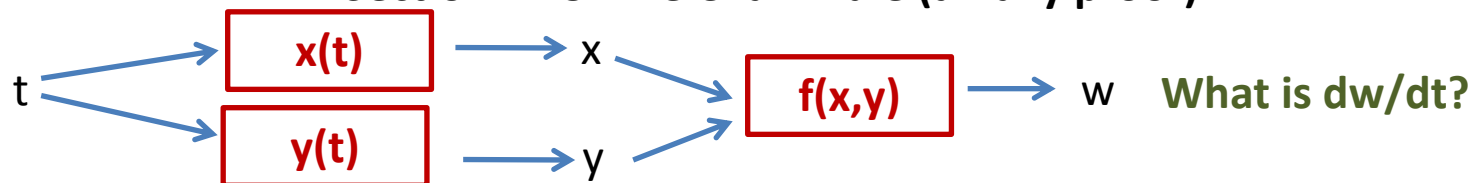
Section 14.5: The Chain Rule

Your turn: $w(x,y)=xy^3 - x^2y$ $x(t) = t^2 + 1$ and $y(t) = t^2 - 1$ Find $\frac{dw}{dt}$

Solution:

$$\begin{aligned}
 \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\
 &= (y^3 - 2xy)(2t) + (3xy^2 - x^2)(2t) \\
 &= ((t^2 - 1)^3 - 2(t^2 + 1)(t^2 - 1))(2t) \\
 &\quad + (3(t^2 + 1)(t^2 - 1)^2 - (t^2 + 1)^2)(2t)
 \end{aligned}$$

Section 14.5: The Chain Rule (a flaky proof)



Claim: $\frac{dw}{dt}$ = Change in output w as input t changes = $\frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$

Proof: Step 1: Recall our tangent approximation plane to $f(x,y)$ at an input point (a,b) :

$$f(x, y) = f(a, b) + \left. \frac{\partial f}{\partial x} \right|_{a,b} (x-a) + \left. \frac{\partial f}{\partial y} \right|_{a,b} (y-b) \leftarrow \text{Equation for plane tangent to } f(a, b)$$

Step 2: So we can call $w = f(x,y)$, $\Delta x = x-a$, $\Delta y = y-b$

Step 3: Rewriting, we have

$$\begin{aligned} w(x, y) &= w(a, b) + \frac{\partial w}{\partial x} (\Delta x) + \frac{\partial w}{\partial y} (\Delta y) \\ w(x, y) - w(a, b) &= \frac{\partial w}{\partial x} (\Delta x) + \frac{\partial w}{\partial y} (\Delta y) \end{aligned} \quad \Rightarrow \quad \Delta w = \frac{\partial w}{\partial x} (\Delta x) + \frac{\partial w}{\partial y} (\Delta y)$$

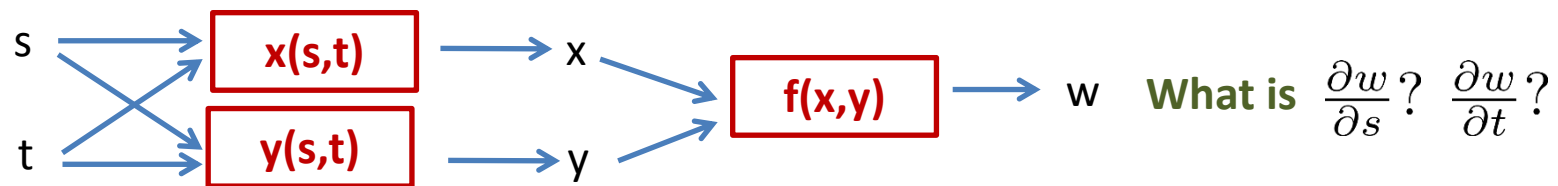
Step 4: Divide both sides by Δt $\frac{\Delta w}{\Delta t} = \frac{\partial w}{\partial x} \frac{\Delta x}{\Delta t} + \frac{\partial w}{\partial y} \frac{\Delta y}{\Delta t}$

Step 5: Take the limit as Δt goes to zero: $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$

This is not much of a proof—but gives the idea of why the terms are added

Section 14.5: The Chain Rule (onwards!)

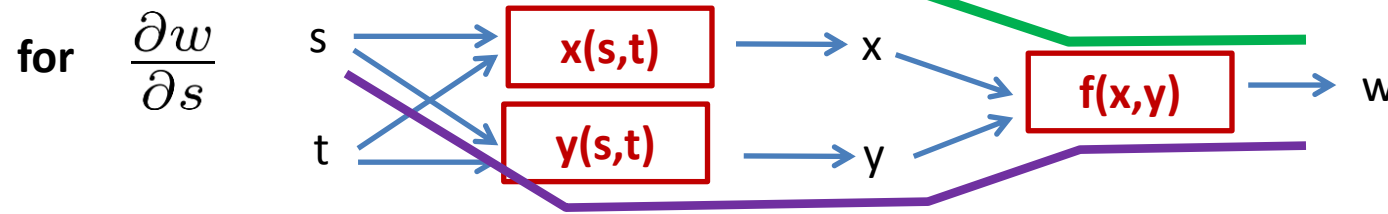
A more complicated case



Remember:

$\frac{\partial w}{\partial s}$ is “how output w changes as input s changes, holding all other inputs constant”

Step 1: Find all paths through the diagram:



Path contributions

path $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s}$

path $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}$

Add the paths $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}$

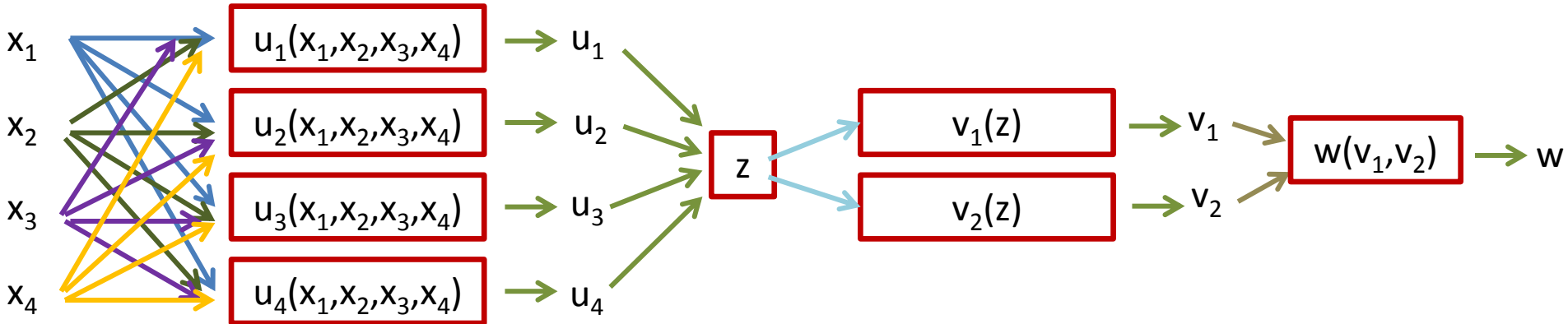
Similarly $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t}$

Section 14.5: The Chain Rule (your turn)

If $w(x,y) = e^x \sin y$ and $x = st^2$ and $y = s^2t$ find $\frac{\partial w}{\partial s}$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} \\ \frac{\partial w}{\partial s} &= (e^x \sin y)(t^2) + (e^x \cos y)(2st) \\ &= (e^{st^2} \sin(s^2t))(t^2) + (e^{st^2} \cos(s^2t))(2st)\end{aligned}$$

Section 14.5: The Chain Rule (one more for you)



What is $\frac{\partial w}{\partial x_3}$?

Again—evaluate all possible paths:

$$\frac{\partial w}{\partial x_3} = \frac{dw}{dz} \frac{\partial z}{\partial x_3}$$

$$= \left[\frac{\partial w}{\partial v_1} \frac{dv_1}{dz} + \frac{\partial w}{\partial v_2} \frac{dv_2}{dz} \right] \left[\frac{\partial z}{\partial u_1} \frac{\partial u_1}{\partial x_3} + \frac{\partial z}{\partial u_2} \frac{\partial u_2}{\partial x_3} + \frac{\partial z}{\partial u_3} \frac{\partial u_3}{\partial x_3} + \frac{\partial z}{\partial u_4} \frac{\partial u_4}{\partial x_3} \right]$$