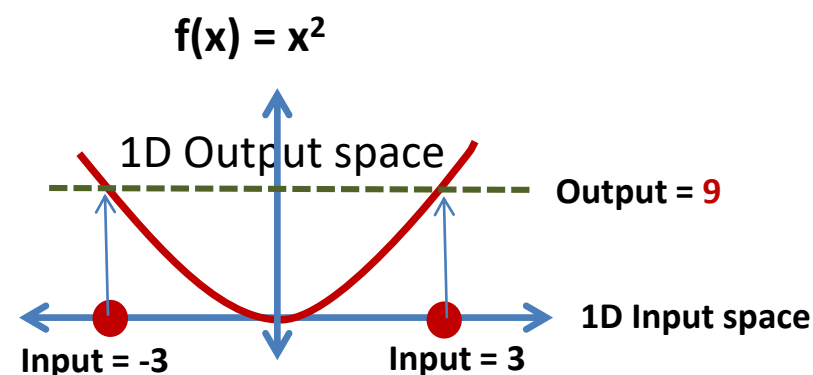
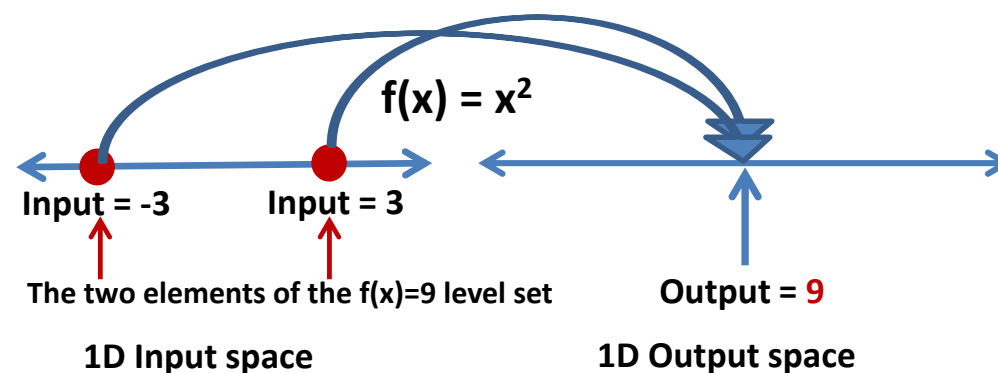


Section 14.1: Review of the Idea of Level Sets

Definition : Given $f(x)$, we call the “ k ” level set the set of all inputs that get sent to the output value of k .

The level set lies in input space



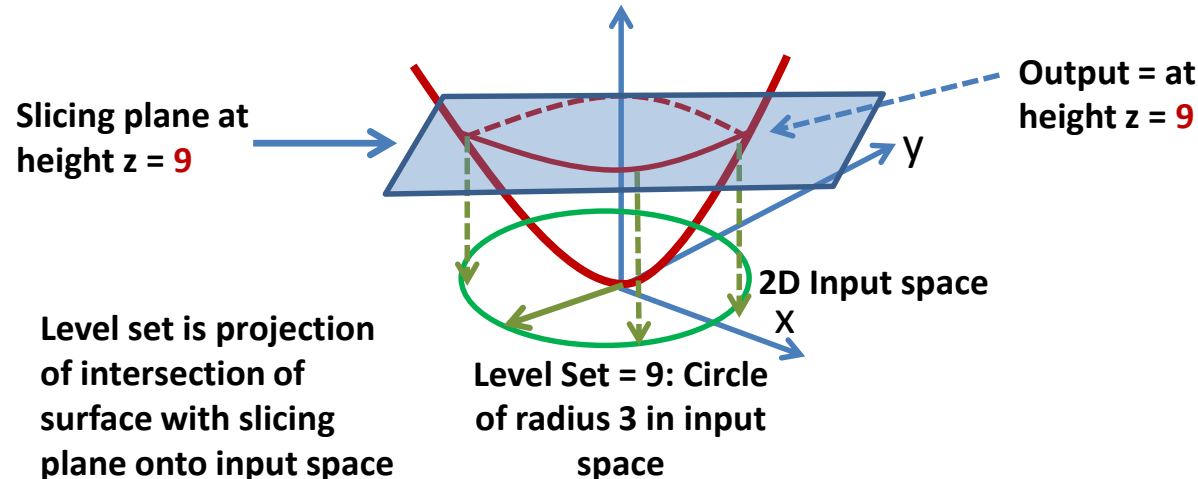
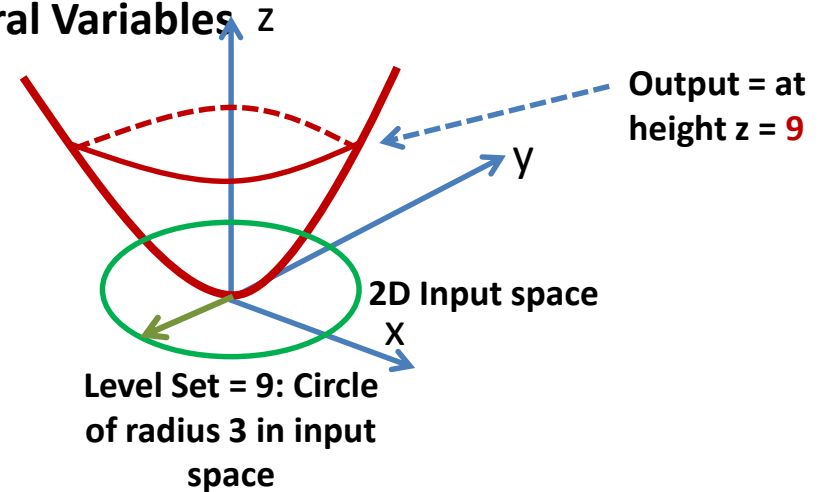
Section 14.1: Functions of Several Variables z

$f(x,y) = x^2 + y^2$ What is the level set $f(x,y) = 9$?

Analytic answer:

it is the set of all input points that get sent to the output 9

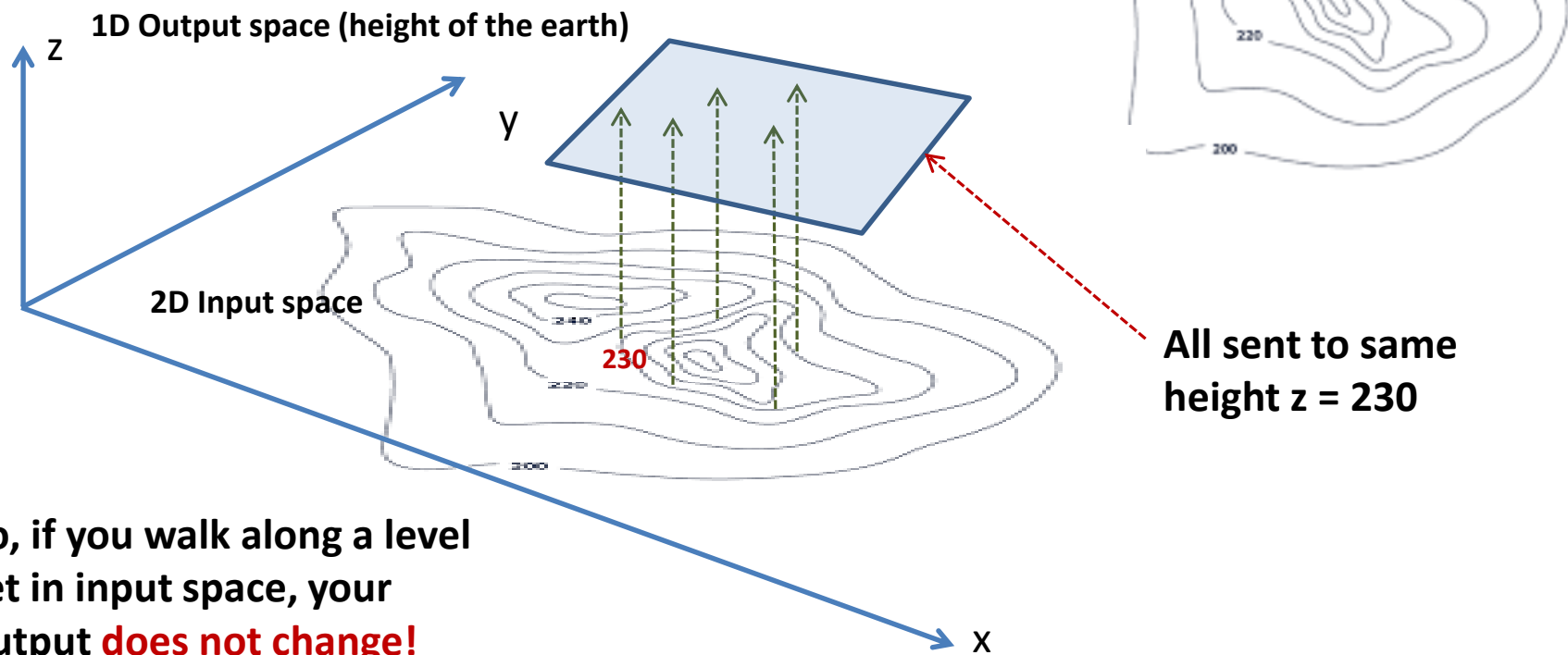
Geometric answer: if you slice the graph with a plane at height k , and project the result down to the input plane, the points you get are the k level set.



Section 14.1: Functions of Several Variables

And that's what a topographic map is!

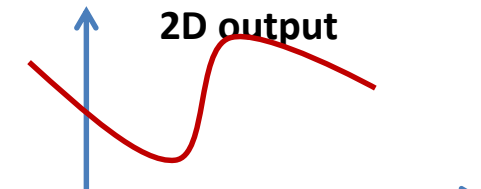
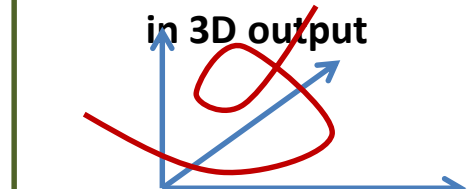
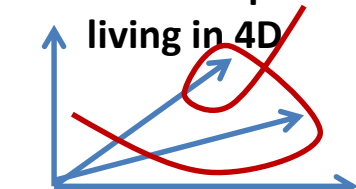
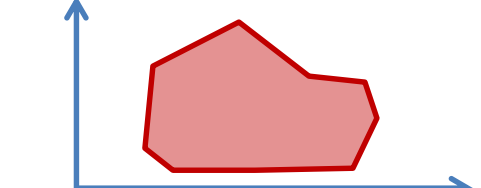
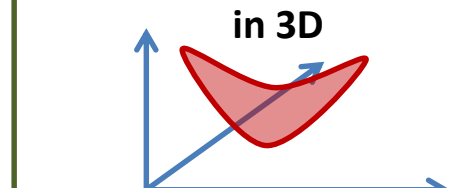
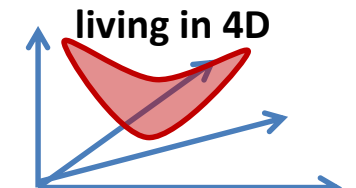
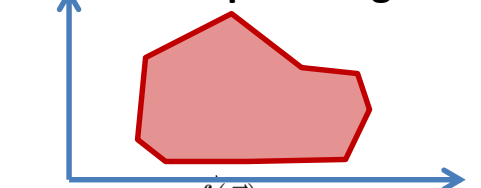
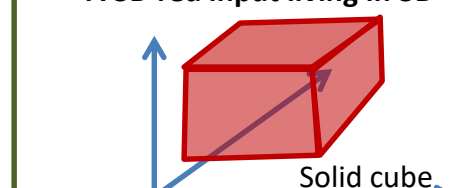
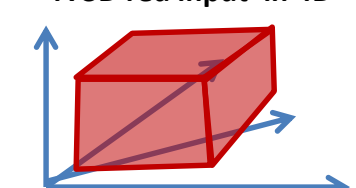
A topographic gives you different k -level sets in the input space, showing what the height of the earth would be. Contour lines show points of the same elevation.



So, if you walk along a level set in input space, your output **does not change!**

Topo map from <https://datavizproject.com/data-type/topographic-map/>

Let’s make sure we agree about what we mean about **dimension**

		Dimension of output space			
		1D output space	2D output space	3D output space	4D output space
Dimension of input space	1D input space	A 1D red input living in 1D output  $f(x) \equiv f(x) : \mathbb{R}^1 \rightarrow \mathbb{R}^1$	A 1D red input living in 2D output  $\vec{f}(x) \equiv (f(x), g(x)) : \mathbb{R}^1 \rightarrow \mathbb{R}^2$	A 1D red input living in 3D output  $\vec{f}(x) \equiv (f(x), g(x), h(x)) : \mathbb{R}^1 \rightarrow \mathbb{R}^3$	A 1D red input living in 4D  $\vec{f}(x) \equiv (f(x), g(x), h(x), i(x)) : \mathbb{R}^1 \rightarrow \mathbb{R}^4$
	2D input space	A 2D red input living in 1D  $f(\vec{x}) \equiv f(x_1, x_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^1$	A 2D red input living in 2D  $\vec{f}(\vec{x}) \equiv (f(x_1, x_2), g(x_1, x_2)) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$	A 2D red input living in 3D  $\vec{f}(\vec{x}) \equiv (f(x_1, x_2), g(x_1, x_2), h(x_1, x_2)) : \mathbb{R}^2 \rightarrow \mathbb{R}^3$	A 2D red input living in 4D  $\vec{f}(\vec{x}) \equiv (f(x_1, x_2), g(x_1, x_2), h(x_1, x_2), i(x_1, x_2)) : \mathbb{R}^2 \rightarrow \mathbb{R}^4$
	3D input space	A 3D red input living in 1D  $f(\vec{x}) \equiv f(x_1, x_2, x_3) : \mathbb{R}^3 \rightarrow \mathbb{R}^1$	A 3D red input living in 2D  $\vec{f}(\vec{x}) \equiv (f(x_1, x_2, x_3), g(x_1, x_2, x_3)) : \mathbb{R}^3 \rightarrow \mathbb{R}^2$	A 3D red input living in 3D  $\vec{f}(\vec{x}) \equiv (f(x_1, x_2, x_3), g(x_1, x_2, x_3), h(x_1, x_2, x_3)) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$	A 3D red input in 4D  $\vec{f}(\vec{x}) \equiv (f(x_1, x_2, x_3), g(x_1, x_2, x_3), h(x_1, x_2, x_3), i(x_1, x_2, x_3)) : \mathbb{R}^3 \rightarrow \mathbb{R}^4$

Section 14.1: Functions of Several Variables

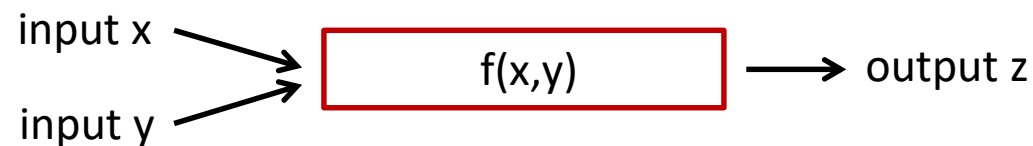
Another example: $f(x,y) = x^2 + 4y^2$ This is a function from 2D to 1D

$$f(\vec{x}) \equiv f(x,y) = z : \mathbb{R}^2 \rightarrow \mathbb{R}^1$$

There are two free input variables:

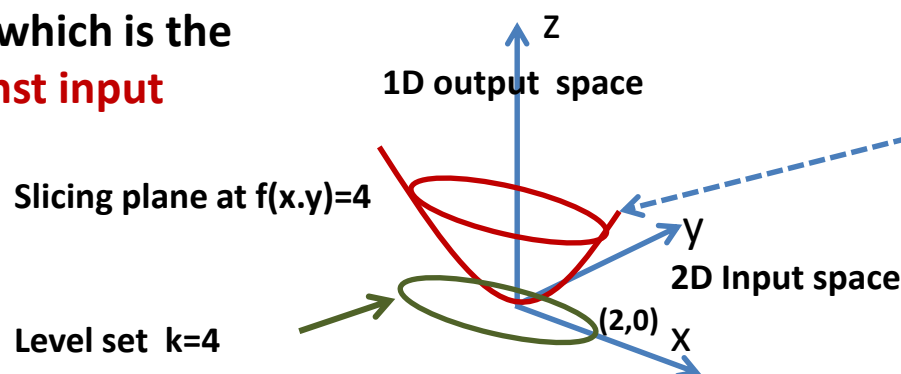
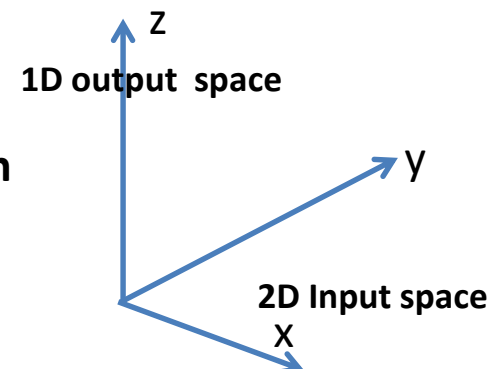
x and y ,

which together live in two-dimensional input space



We can plot the output using a third dimension

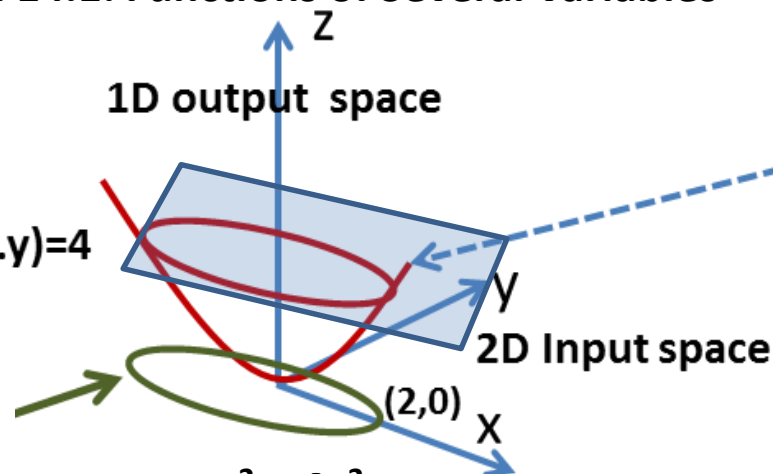
and then use the equation to
link output space to input space,
generating a surface which is the
graph of output against input



$$z = f(x,y) = x^2 + 4y^2$$

(this is supposed to be an elliptical surface!)

Section 14.1: Functions of Several Variables

The level set $k=4$ 

$$z = f(x, y) = x^2 + 4y^2$$

(this is supposed to be an elliptical surface!)

Level set $k=4$
 = Curve in x-y plane satisfying $4 = x^2 + 4y^2$

Let's really talk about *dimensions*

For a mapping $f(\vec{x}) \equiv f(x, y) = z : \mathbb{R}^2 \rightarrow \mathbb{R}^1$ **from 2D input to 1D output**

- It requires three dimensions to contain the graph of output against input
- The graph *itself* is a 2D surface living in 3D space
- The level set is a 1D curve in 2D input space
- Again—the level curve satisfying $4 = x^2 + 4y^2$ is 1D curve, because it only has 1 free variable (as x changes, that nails down y).

Again, for a mapping $f(\vec{x}) \equiv f(x, y) = z : \mathbb{R}^2 \rightarrow \mathbb{R}^1$

We graph it in 3D. The thing itself is 2D. The level set is 1D.

Section 14.1: Functions of Several Variables

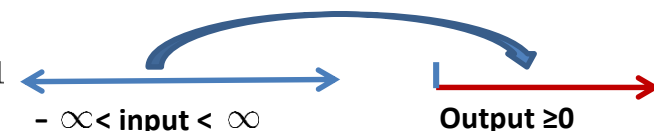
Again, for a mapping $f(\vec{x}) \equiv f(x, y) = z : \mathbb{R}^2 \rightarrow \mathbb{R}^1$

We graph it in 3D. The thing itself is 2D. The level set is 1D.

Let's go down a dimension to see if this is still true:

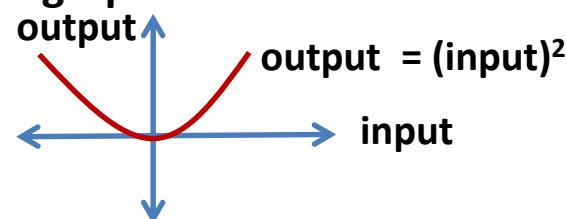
Example: $f(x) = x^2$

(1) This is mapping from 1D to 1D $f(x) \equiv x^2 : \mathbb{R}^1 \rightarrow \mathbb{R}^1$



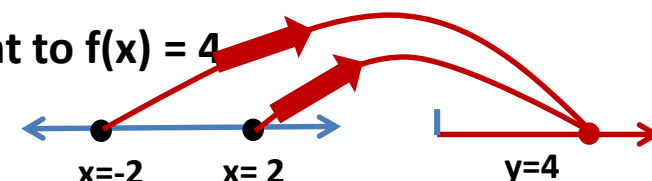
(2) There is one free input variable, so the graph itself is 1D.

(3) We can add a second dimension y and graph it in $\mathbb{R}^2$



(4) The level set $k=4$ is the set of all points that get sent to $f(x) = 4$

Those two points are zero-dimensional



So again: **We graph it in 2D. The thing itself is 1D. The level set is 0D.**

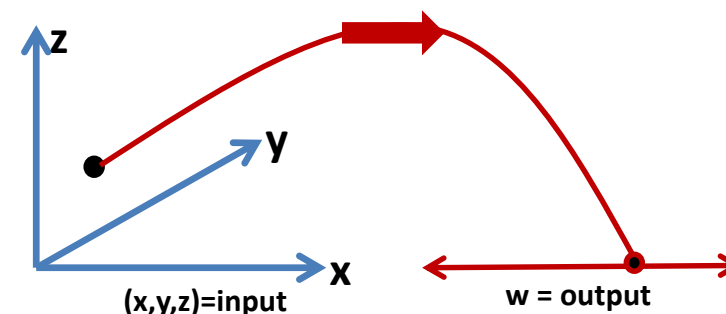
Section 14.1: Functions of Several Variables

Okay—now I am getting excited—let's go **up** a dimension

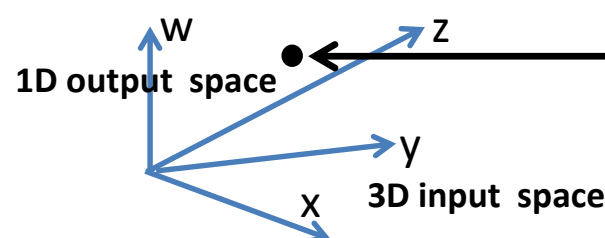
Example: $f(x,y,z) = x^2 + y^2 + z^2$

(1) This is a 3D function

$$f(\vec{x}) \equiv f(x, y, z) = w : \mathbb{R}^3 \rightarrow \mathbb{R}^1$$



(2) We need a fourth dimension to graph it



This point is part of the **graph** of the 3D surface $(x,y,z,f(x,y,z))$ which lives in 4D space

(3) And the k level set is the set of all points (x,y,z) such that $k = f(x,y,z) = x^2 + y^2 + z^2$ which is a sphere of radius $\sqrt{k}$

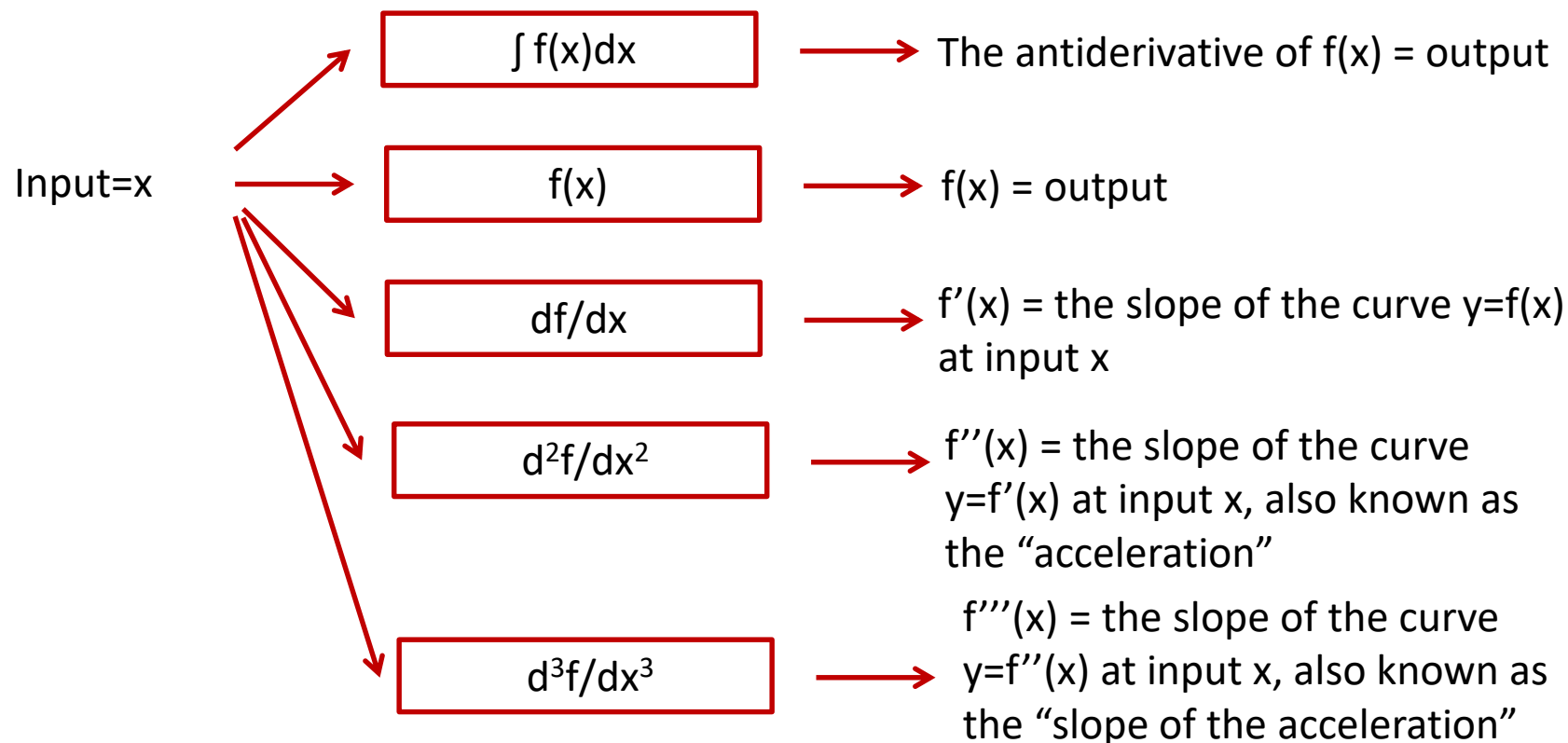
(4) this is a 2D object, since there are only 2 free variables

So again: **We graph it in 4D. The thing itself is 3D. The level set is 2D.**

Read Section 14.2 on your own---Section 14.3: On to derivatives!

Definition: First, let's remember 1D calculus: Given $f(x)$, we define the derivative of $f(x)$ with respect to x

$$\frac{df}{dx} \equiv f' \equiv \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$



Section 14.3: On to derivatives--Multivariables!

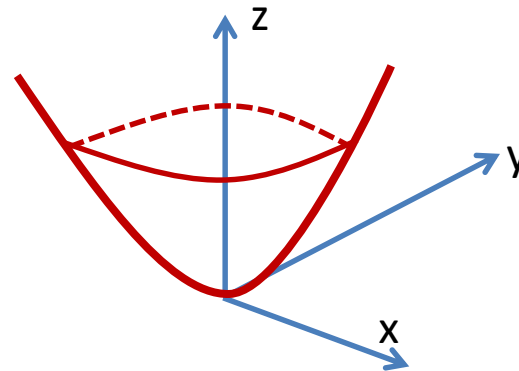
Definition: Given $f(x,y)$, we **define** the partial derivative of $f(x,y)$ with respect to x as the **derivative of f as x changes, assuming y is constant.**

Notation: partial derivative $\frac{\partial f(x,y)}{\partial x} \equiv \frac{\partial f}{\partial x} \equiv f_x(x,y)$

Example: $f(x,y) = x^2 + 3y^2$

$$f_x = (2x)$$

$$f_y = (6y)$$



Example: $f(x,y) = 12x^3y + x^2 + 3y^2$

$$f_x = (36x^2y + 2x)$$

$$f_y = (12x^3 + 6y)$$

Example: $f(x,y) = \sin(xy)$

$$f_x = y \cos(xy)$$

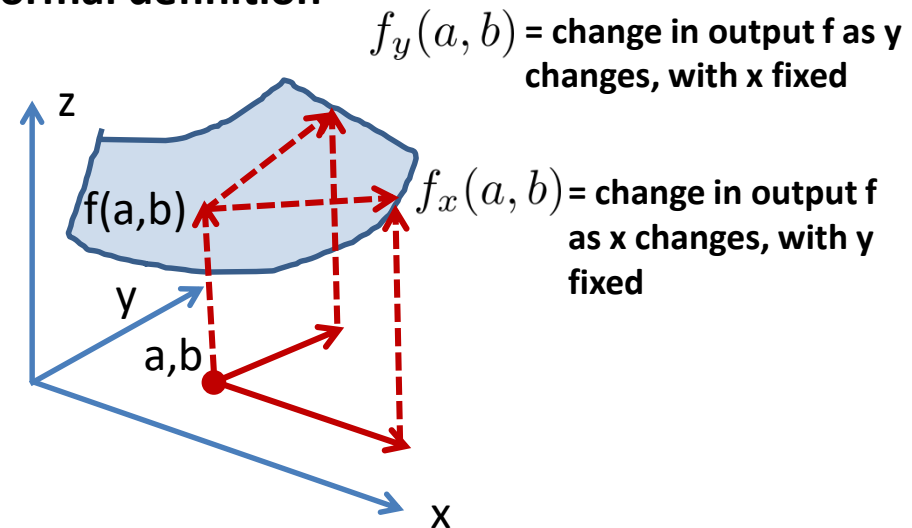
$$f_y = x \cos(xy)$$

Section 14.3: The formal definition

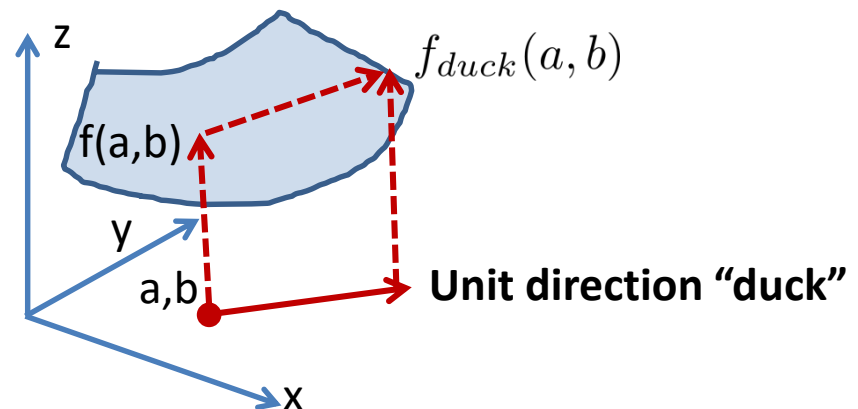
Let's define the partial derivatives formally:

$$f_x(a, b) \equiv \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$$

$$f_y(a, b) \equiv \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}$$



So, you should think of the partial derivative in the “duck” direction as how much the function changes as you move from (a,b) in the unit vector direction “duck”



Section 14.3: The formal definition

More examples: your turn $f(x,y,z) = x y^2 + x y z^3 + \cos(xyz)$

Find f_x, f_y, f_z

Solution:

$$f_x = y^2 + yz^3 - yz \sin xyz$$

$$f_y = 2xy + xz^3 - xz \sin xyz$$

$$f_z = 3xyz^2 - xy \sin xyz$$

It seems obvious we can keep going: $f_{xx} \equiv \frac{\partial^2 f}{\partial x^2} = \frac{\partial(\frac{\partial f}{\partial x})}{\partial x}$

Find f_{xx}, f_{yy}, f_{zz}

$$f_{xx} = z \sin xyz - (yz)(yz) \cos(xyz)$$

$$f_{yy} = 2x + -xz(xz) \cos(xyz)$$

$$f_{zz} = 6xyz - xy(xy) \cos xyz$$

Section 14.3: We can also do “cross-derivatives”

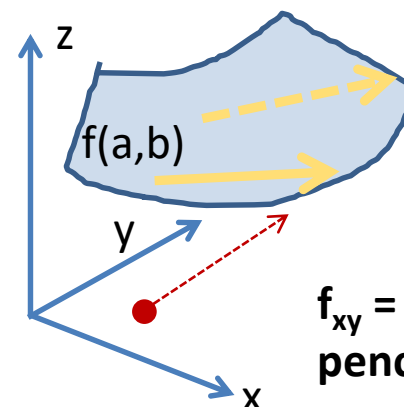
$$f_{xy} \equiv \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial \left(\frac{\partial f}{\partial x} \right)}{\partial y}$$

The rate of change as y changes of the rate of change of f as x changes

To see this visually, imagine a pencil tangent to a surface in a direction where y is constant. As that pencil is translated, always pointing in the xy plane, its change in slope is given by f_{xy}

Example: $f(x, y) = x^2 y^3 + e^{xy^4}$

Find f_x, f_y, f_{xy}, f_{yx}



f_{xy} = change in pencil slope as you move in the y direction

$$f_x = 2xy^3 + y^4 e^{xy^4}$$

$$f_{xy} = 6xy^2 + 4y^3 e^{xy^4} + y^4 (x4y^3) e^{xy^4}$$

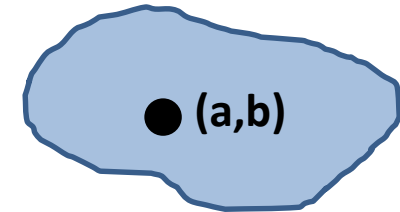
$$f_y = x^2(3y^2) + (4xy^3) e^{xy^4}$$

$$f_{yx} = 2x(3y^2) + (4y^3) e^{xy^4} + (4xy^3)(y^4) e^{xy^4}$$

Wow! Why are these the same?

Section 14.3: Clairaut's Theorem

Suppose $f(x,y)$ is defined on a disk around a point (a,b) , and that f_{xy} and f_{yx} are both continuous in that disk. Then



$f_{xy} = f_{yx}$ Often referred to as the "Equality of Mixed Partial"

Question: Suppose $f(x, y) = x^3 e^{x^3} \sin(xe^{xy} (x^y)^x)$

What is $f_{xy} - f_{yx}$?

Solution: By Clairaut's theorem, the answer is zero!

I won't prove this ---but try looking it up yourself to see how it is done—it basically depends on carefully analyzing the limits

$$f_{xy} = \lim_{h_2 \rightarrow 0} \frac{\lim_{h_1 \rightarrow 0} \frac{f(x+h_1, y+h_2) - f(x, y+h_2)}{h_1} - \lim_{h_1 \rightarrow 0} \frac{f(x+h_1, y) - f(x, y)}{h_1}}{h_2}$$

$$f_{yx} = \lim_{h_1 \rightarrow 0} \frac{\lim_{h_2 \rightarrow 0} \frac{f(x+h_1, y+h_2) - f(x, y+h_2)}{h_2} - \lim_{h_2 \rightarrow 0} \frac{f(x+h_1, y) - f(x, y)}{h_2}}{h_1}$$

And showing that they are the same (now you see why continuity in f_{xy} and f_{yx} is needed)

(I will come back to 14.4—but first): Section 14.5: Implicit Relationships

Suppose $x + 2y = 6$. Then x and y are not independent: x and y are forced to cooperate

This is called an **“implicit relationship”**

And we could ask “how does y change when x changes? This is what is meant by dy/dx

Question: Suppose $x^3 + y^3 + z^3 + 6xyz = 1$

This is an **implicit relationship** between x, y , and z . x, y , and z are linked So what is $\frac{\partial z}{\partial x}$?

Commentary:

(1) What does this question even mean?

(2) It means, how does z change when x changes,

assuming y is held fixed (that’s what a partial derivative means!) ?

Let’s do it:

Step 1: Take the partial derivative of both sides with respect to x , remembering that y is held fixed, and z is now a function of x

$$\frac{\partial}{\partial x} (x^3 + y^3 + z^3 + 6xyz) = \frac{\partial}{\partial x} (1)$$

$$3x^2 + 0 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0$$

Step 2: Solve for $\frac{\partial z}{\partial x}$

$$3x^2 + 6yz + (3z^2 + 6xy) \frac{\partial z}{\partial x} = 0 \rightarrow \frac{\partial z}{\partial x} = -\frac{3x^2 + 6yz}{3z^2 + 6xy}$$

Section 14.5: Implicit Relationships

Let’s use our function boxes to make sure we know what we are talking about:



We start with no relation between x, y and z

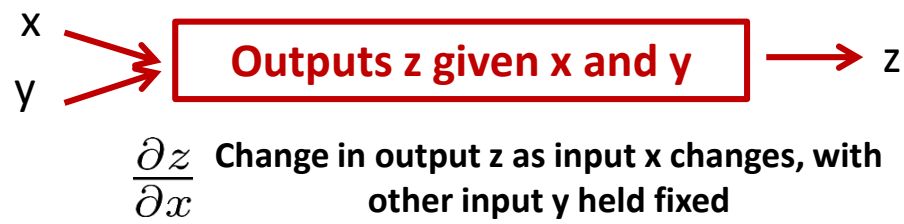
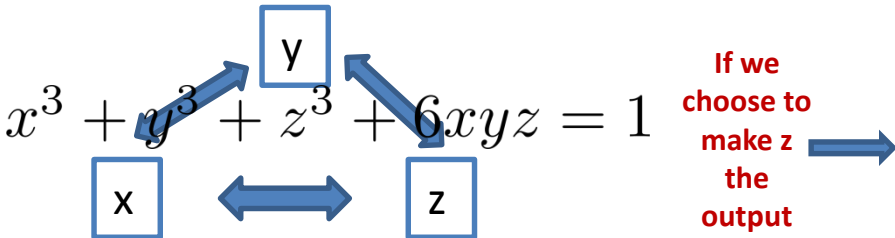


Image from busy.org

(2) We discover there is an equation $x^3 + y^3 + z^3 + 6xyz = 1$ that links them all together:



Image from netclipart.com



And we could have chosen any variable as the output



$\frac{\partial y}{\partial z}$ Change in output y as input z changes, with other input x held fixed

Section 14.3: Implicit Relationships

Your turn: $x^2 + 2y^2 + 3z^2 = 1$ Find $\frac{\partial z}{\partial x}$

Solution: Step 1: take the partial derivative of both sides with respect to x (holding y fixed)

$$\frac{\partial}{\partial x}(x^2 + 2y^2 + 3z^2) = \frac{\partial}{\partial x}(1)$$

$$2x + 0 + 6z \frac{\partial z}{\partial x} = 0$$

Step 2: Solve for $\frac{\partial z}{\partial x} \rightarrow \frac{\partial z}{\partial x} = (-2x)/(6z)$

Your turn: $e^z = xyz$ Find $\frac{\partial y}{\partial x}$

Solution: Step 1: take the partial derivative of both sides with respect to x (holding z fixed)

$$\frac{\partial}{\partial x}(e^z) = \frac{\partial}{\partial x}(xyz) \quad (\text{Remember: } y = y(x) \text{ and } z \text{ is fixed—use product rule})$$

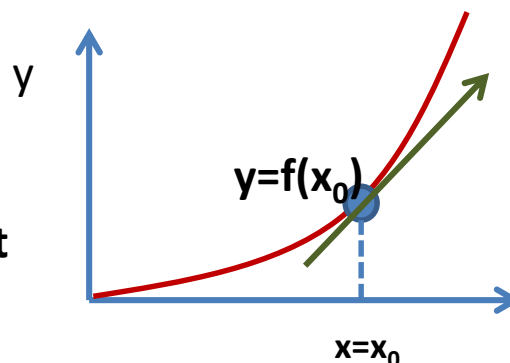
$$0 = yz + x \frac{\partial y}{\partial x} z$$

Step 2: Solve for $\frac{\partial y}{\partial x} \rightarrow \frac{\partial y}{\partial x} = (-yz)/(xz) = -y/x$

Section 14.4: Tangent Planes and linear approximations

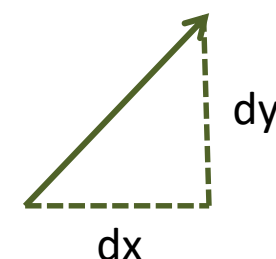
Let's recall 1D Calculus: $y=f(x)$

Tangent line at $(x_0, f(x_0))$ touches the graph $y=f(x_0)$ at only one point in a near $(x_0, f(x_0))$



Tangent line with slope $f'(x_0)$ going through the point $(x_0, f(x_0))$

The tangent line is given by $y - f(x_0) = \text{slope}(x - x_0) = \left. \frac{df}{dx} \right|_{x_0} (x - x_0)$

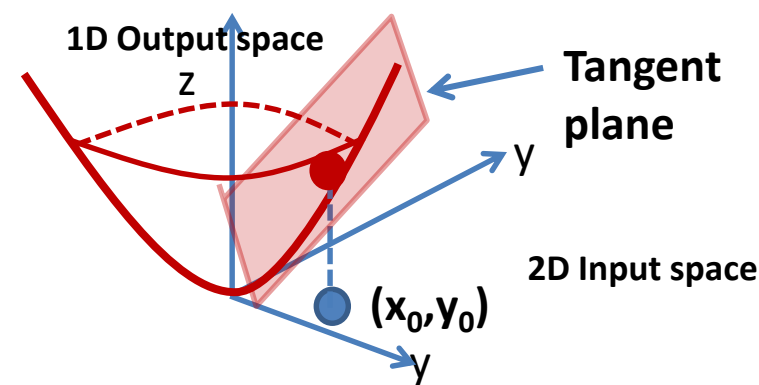


Tangent vector = $(dx, dy) = (1, dy/dx) = (1, f'(a))$

We want to construct a similar idea for functions of two (or more variables):

The Tangent Plane

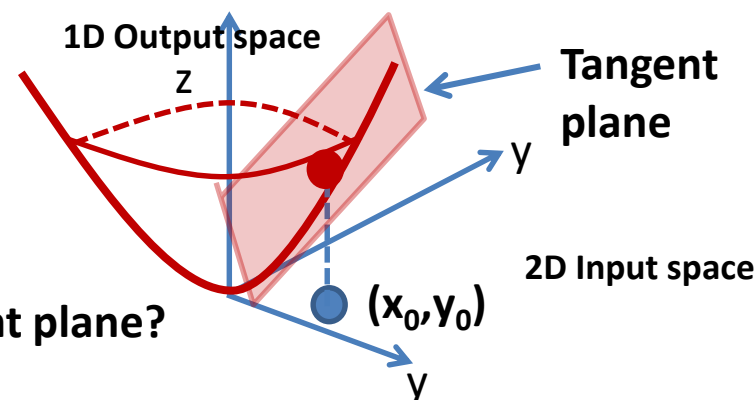
Tangent plane at $(x_0, y_0, f(x_0, y_0))$ touches the graph $z=f(x_0, y_0)$ at only one point in a near $(x_0, y_0, f(x_0, y_0))$



Section 14.4: Tangent Planes and linear approximations

Tangent plane at $(x_0, y_0, f(x_0, y_0))$ touches the graph $z=f(x_0, y_0)$ at only one point in a near $(x_0, y_0, f(x_0, y_0))$

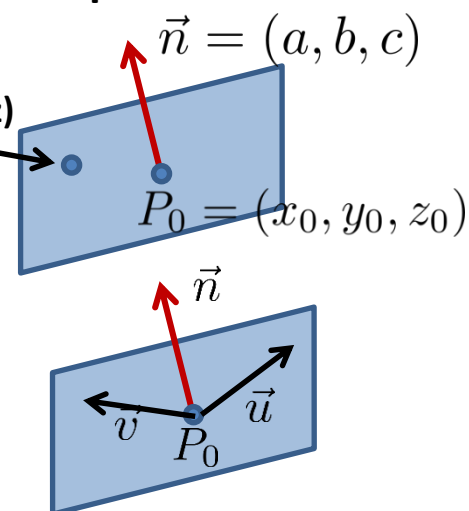
How are we going to find an equation for the tangent plane?



Idea #1! Do you remember we had a formula for a plane going through the point (x_0, y_0, z_0) with normal vector (a, b, c) ?

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Arbitrary point (x, y, z) on plane



Realization 1: we know the point P_0 where the tangent plane touches the surface: $P_0 = (x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$

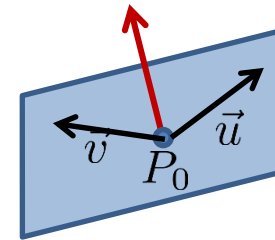
Realization 2: if we had two vectors $\vec{u}$ and $\vec{v}$ in the tangent plane, we could take their cross product to find the normal $\vec{n} = \vec{u} \times \vec{v}$

Section 14.4: Tangent Planes and linear approximations

Formula for a plane going through the point

(x_0, y_0, z_0) with normal vector (a, b, c)

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$



Tangent point is $P_0 = (x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$

How can we find two vectors in the tangent plane?

Idea #2: We can slice the graph of $z=f(x,y)$ with a plane $y=y_0$

This gives a purple curve whose y coordinate never changes and lies on the surface.

Tangent vector with slope $\left. \frac{df}{dx} \right|_{x_0, y_0}$

So this purple curve is the graph of $(x, y_0, f(x, y_0))$

And the x partial derivative of this purple curve at (x_0, y_0) gives the slope $\left. \frac{\partial f}{\partial x} \right|_{x_0, y_0}$ of the tangent vector

at P_0 lying in the slicing plane

So one tangent vector is

$$\vec{v} = \left(1, 0, \left. \frac{\partial f}{\partial x} \right|_{x_0, y_0} \right)$$

Section 14.4: Tangent Planes and linear approximations

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Tangent point is $P_0 = (x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$

Obtain another tangent vector by slicing the graph of $z=f(x,y)$ with a plane $x=x_0$

This gives a purple curve whose x coordinate never changes and lies on the surface.

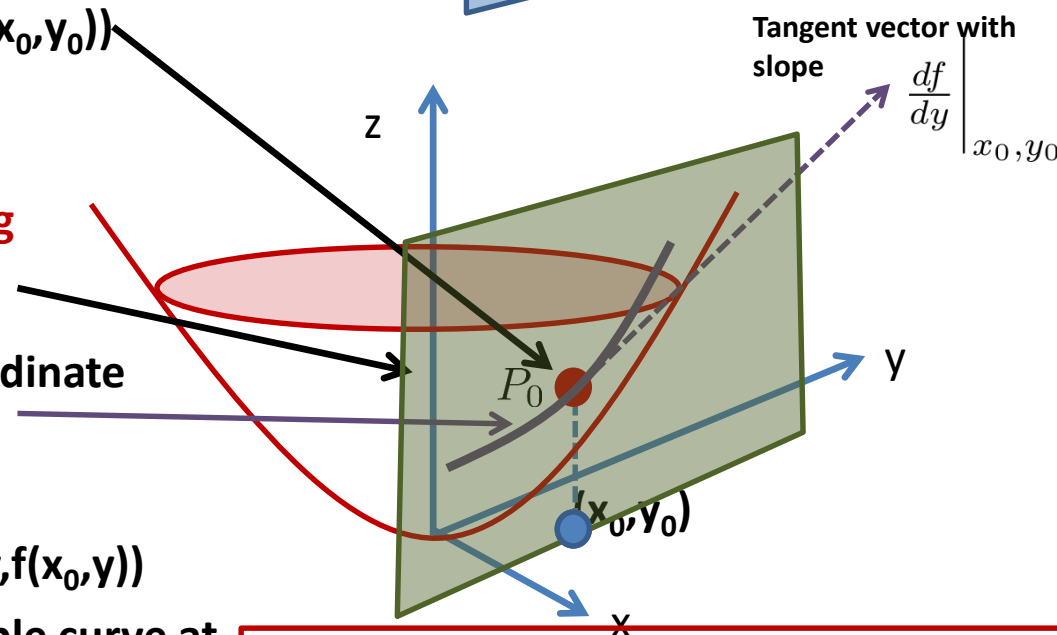
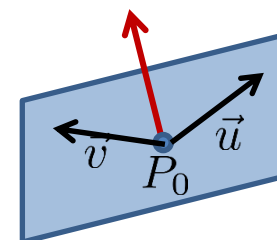
So this purple curve is the graph of $(x_0, y, f(x_0, y))$

And the y partial derivative of this purple curve at (x_0, y_0) gives the slope $\left. \frac{\partial f}{\partial y} \right|_{x_0, y_0}$ of the tangent vector

at P_0 lying in the slicing plane

So one tangent vector is

$$\vec{v} = \left(0, 1, \left. \frac{\partial f}{\partial y} \right|_{x_0, y_0} \right)$$



Section 14.4: Tangent Planes and linear approximations

One tangent vector is $\vec{u} = (1, 0, \left. \frac{\partial f}{\partial x} \right|_{x_0, y_0})$

One tangent vector is $\vec{v} = (0, 1, \left. \frac{\partial f}{\partial y} \right|_{x_0, y_0})$

$$\vec{n} = (a, b, c) = \vec{u} \times \vec{v} = (1, 0, \left. \frac{\partial f}{\partial x} \right|_{x_0, y_0}) \times (0, 1, \left. \frac{\partial f}{\partial y} \right|_{x_0, y_0}) = (-\left. \frac{\partial f}{\partial x} \right|_{x_0, y_0}, -\left. \frac{\partial f}{\partial y} \right|_{x_0, y_0}, 1) \leftarrow \text{check this!}$$

Back to our formula for a plane:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

(I temporarily
stopped writing $\left. \right|_{x_0, y_0}$)

Substitute everybody in:

$$-\left. \frac{\partial f}{\partial x} \right|_{x_0, y_0} (x - x_0) + -\left. \frac{\partial f}{\partial y} \right|_{x_0, y_0} (y - y_0) + 1(z - f(x_0, y_0)) = 0$$

Solve for z (remembering that $z_0 = f(x_0, y_0)$):

$$z - f(x_0, y_0) = \left. \frac{\partial f}{\partial x} \right|_{x_0, y_0} (x - x_0) + \left. \frac{\partial f}{\partial y} \right|_{x_0, y_0} (y - y_0)$$

**Formula for the
tangent plane**

Section 14.4: Tangent Planes and linear approximations

**Formula for the
tangent plane**

$$z - f(x_0, y_0) = \left. \frac{\partial f}{\partial x} \right|_{x_0, y_0} (x - x_0) + \left. \frac{\partial f}{\partial y} \right|_{x_0, y_0} (y - y_0)$$

**Notice how much it looks like
our formula for the slope of a
tangent line:**

$$y - f(x_0) = \text{slope}(x - x_0) = \left. \frac{df}{dx} \right|_{x_0} (x - x_0)$$

Example: Find the equation for the plane tangent to $z = 2x^2 + y^2$ at the input point $x_0=1$ $y_0=1$

Solution: Step 1: find the point on the surface at the input $x_0=1$ $y_0=1$

$$z_0 = f(x_0, y_0) = f(1, 1) = 2(1)^2 + 1^2 = 3$$

Step 2: find the partial derivatives at the input point:

$$f_x = 4x \text{ so at input } x_0=1 \ y_0=1 \ f_x=4$$

$$f_y = 2y \text{ so at input } x_0=1 \ y_0=1 \ f_y=2$$

Step 3: put them into your equation for the tangent plane: $z-3 = 4(x-1) + 2(y-1)$