

Section 13.1

We have vectors $\vec{u} = (a_1, b_1, c_1, d_1, e_1)$ $\vec{v} = (a_2, b_2, c_2, d_2, e_2)$

$$\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta = \text{a number}$$

and new ways to combine them

$$\vec{u} \times \vec{v} = \text{a funky expression} = \text{a vector}$$

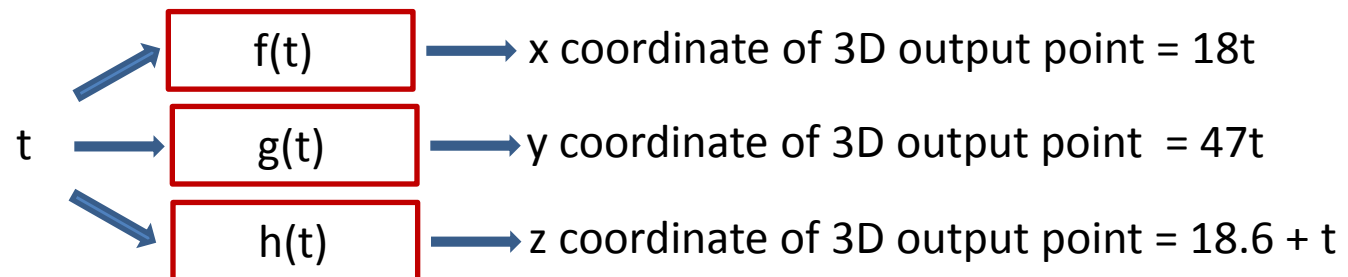
so far, only defined for 3D

$$c\vec{u} = \text{a vector}$$

Now we want to build **vector-valued functions**

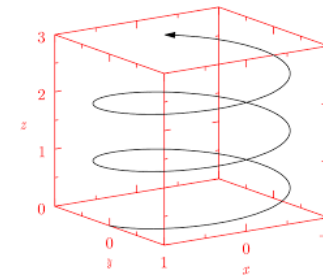
Example: $f(\mathbf{x}) : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ such as $y = f(x) = 6x$ $x \longrightarrow \boxed{f(x)} \longrightarrow y$

Example: $f(\vec{t}) : \mathbb{R}^1 \rightarrow \mathbb{R}^3$ such as $(f(t), g(t), h(t)) = (18t, 47t, 18.6 + t)$



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Example: $x = f(t) = \cos(t)$
 $y = g(t) = \sin(t)$
 $z = h(t) = t$
 $\vec{f}(x) : \mathbb{R}^1 \rightarrow \mathbb{R}^3$



[image from artofproblemsolving.com]

Now, since we have functions, we want to talk about continuity and limits

Definition: the limit as $t \rightarrow a$ of a vector function $\vec{f}(t)$ is the vector made up of the Individual, component-wise, limits:

$$\lim_{t \rightarrow 8} (t, t^2, t^{1/3}) = \left(\lim_{t \rightarrow 8} t, \lim_{t \rightarrow 8} t^2, \lim_{t \rightarrow 8} t^{1/3} \right) = (8, 64, 2),$$

Realization: We can think of any vector-valued function

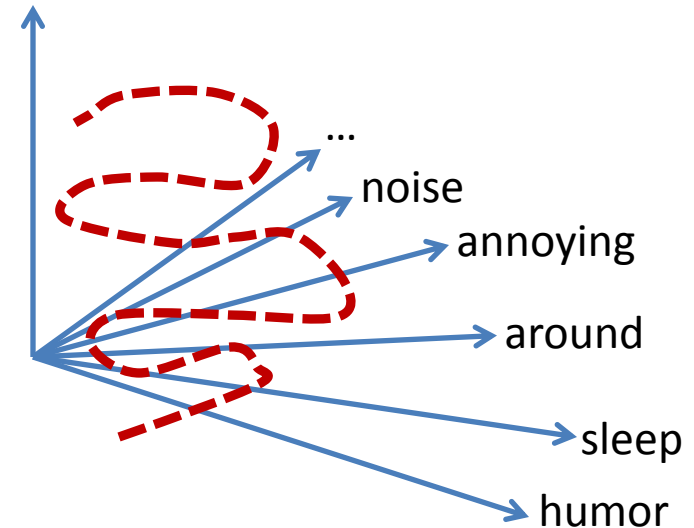
$$\vec{f}(x) : \mathbb{R}^1 \rightarrow \mathbb{R}^{82} = (f_1(x), f_2(x), f_3(x), \dots, f_{82}(x))$$

as the **trajectory of a mosquito in 82-dimensional space....**

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Example: How you currently feel about your roommate (or sister, brother, parents, etc.) is a vector-valued function of time t

Roommate (t) =
(sense of humor (t),
how much they sleep (t),
how often they are there (t),
how annoying their friends are (t),
how much noise they make (t),
.....)



A trajectory through high-dimensional space

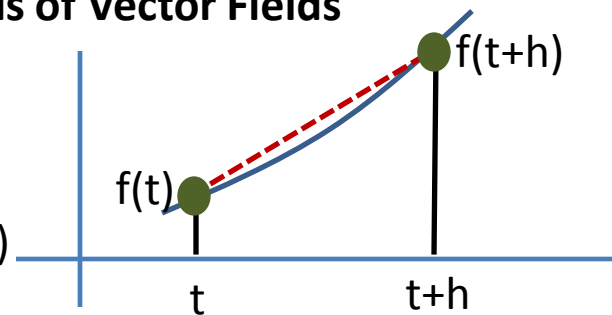
Section 13.2: Derivatives and Integrals of Vector Fields

Remember 1D:

$$f'(t) = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}$$

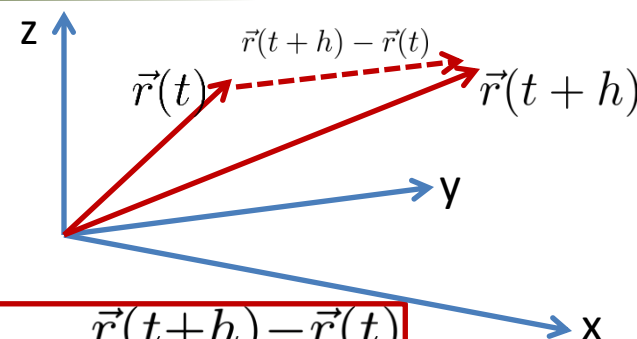
= slope of straight line

connecting $(t, f(t))$ with $(t+h, f(t+h))$



How can we think about this for vector-valued functions?

Definition of vector derivative



$$\frac{d\vec{r}(t)}{dt} = \text{Also written as } \vec{r}' = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Please note: this “derivative” is a **vector !**

Fortunately, it is really easy to compute! $\vec{r}(t) = (f(t), g(t), h(t)) \rightarrow \vec{r}'(t) = (f'(t), g'(t), h'(t))$

Example: if $\vec{r}(t) = (t^2, t^3, t^5)$ find $\vec{r}'$ at $t=2$:

Solution: $\vec{r}' = (2t, 3t^2, 5t^4) = (4, 12, 80)$

Section 13.2: Derivatives and Integrals of Vector Fields

Okay—what about integration of a vector function?

If $\vec{r}(t) = (f(t), g(t), h(t))$ then $\int_a^b \vec{r}(t) dt = \left(\int_a^b f(t) dt, \int_a^b g(t) dt, \int_a^b h(t) dt \right)$

Example: Suppose $\vec{r}(t) = (f(t), g(t), h(t)) = (t, t^2, t^3)$ **Find** $\int_2^3 \vec{r}(t) dt$

Solution:
$$\int_2^3 \vec{r}(t) dt = \left(\int_2^3 t dt, \int_2^3 t^2 dt, \int_2^3 t^3 dt \right) = \left(\left|_2^3 \frac{1}{2} t^2, \left|_2^3 \frac{1}{3} t^3, \left|_2^3 \frac{1}{4} t^4 \right. \right) \right.$$

$$= \left(\frac{1}{2}(3^2 - 2^2), \frac{1}{3}(3^3 - 2^3), \frac{1}{4}(3^4 - 2^4) \right)$$

Yeah, but what does the vector-valued integral **actually mean**?

Answer: $\int_a^b \vec{r}(t) dt = \left(\int_a^b f(t) dt, \int_a^b g(t) dt, \int_a^b h(t) dt \right)$

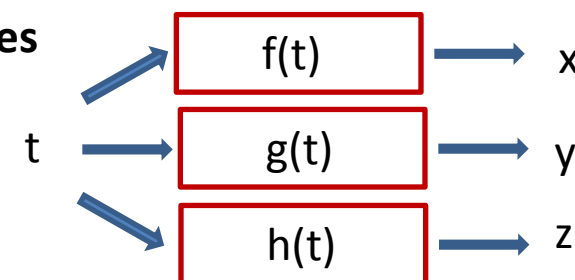
is the total amount of :

f(t) between a < t < b, g(t) between a < t < b, h(t) between a < t < b

Section 14.1: Functions of Several Variables

So far, we have talked about vector-valued functions of one input variable:

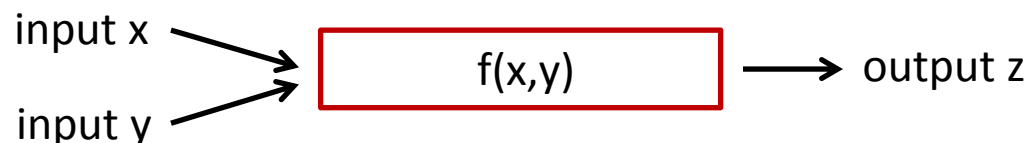
$$f(\vec{t}) : \mathbb{R}^1 \rightarrow \mathbb{R}^3$$



Let's have more fun, and talk about functions of several input variables: $z = f(x, y)$

Definition: A function of two variables is a mapping with two inputs and one output

Example: $z = f(x, y)$

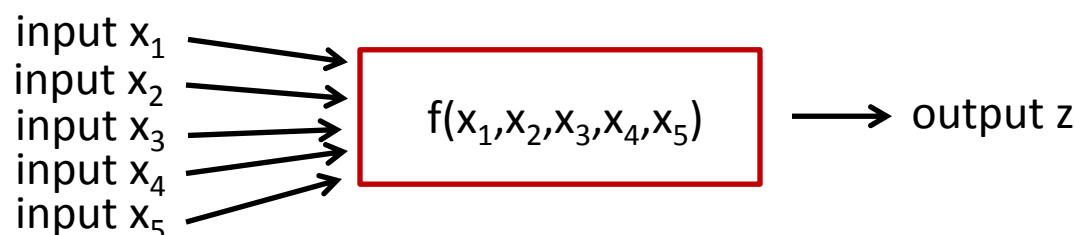


Example:
 $z = f(x, y) = x^2 + 6y$

The notation here is precise, but tricky: I should write: $f(\vec{x}) \equiv f(x, y) = z : \mathbb{R}^2 \rightarrow \mathbb{R}^1$

(the input vector $\vec{x}$ has two components, namely x and y)

Another example: a function of 5 variables $z = f(\vec{x}) \equiv f(x_1, x_2, x_3, x_4, x_5) : \mathbb{R}^5 \rightarrow \mathbb{R}^1$



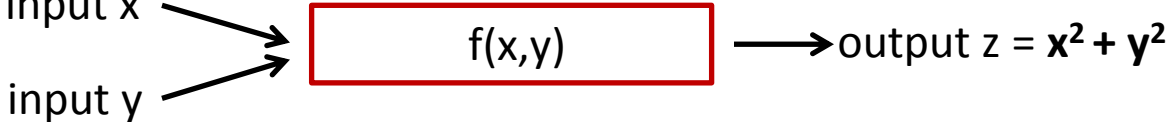
Section 14.1: Functions of Several Variables

And now comes a moment of slipperiness

Question: What is meant by $f(x,y)=x^2 + y^2$?

Important point: **it is giving the output for a function of two input variables x and y**

It is **not a circle.**



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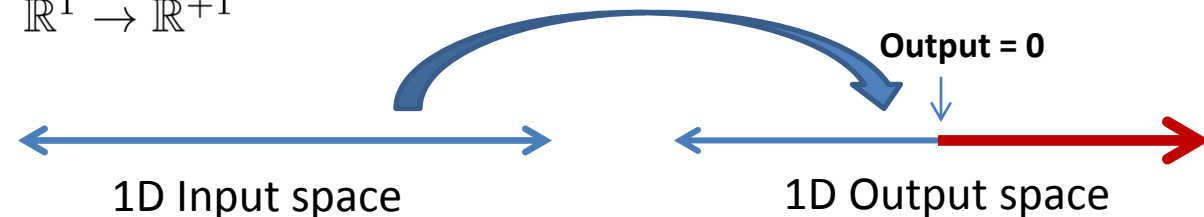
    graph LR
      x[input x] --> f["f(x,y)"]
      y[input y] --> f
      f --> z["output z = x^2 + y^2"]
  
```

Let's really dig into this:

Question: Back to 1D: What is $f(x) = x^2$?

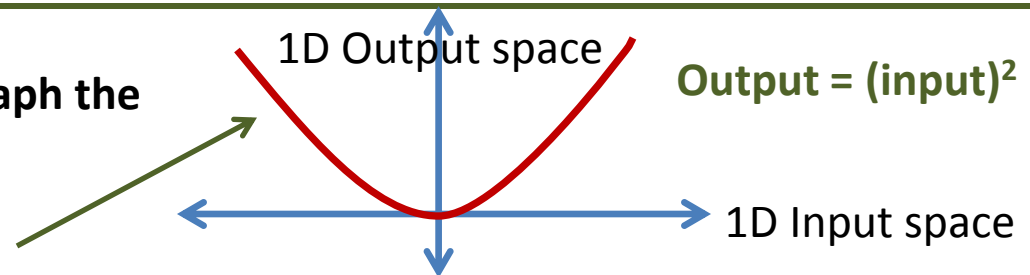
Answer: It is a function: input is any real number and output is non-negative real number

It is a mapping from $\mathbb{R}^1 \rightarrow \mathbb{R}^{+1}$



On the other hand, if I ask you to graph the output of this function vs. the input

Then the graph is a parabola

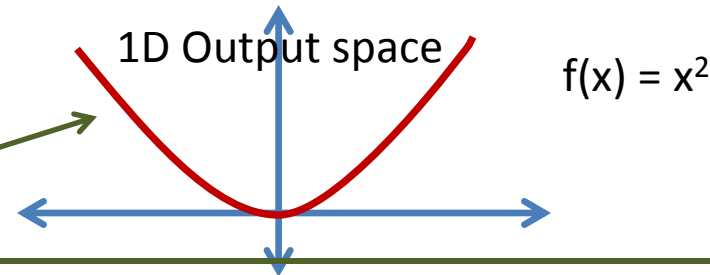


Section 14.1: Functions of Several Variables

I really want to push this issue

If we plot output against input

then the graph is a parabola

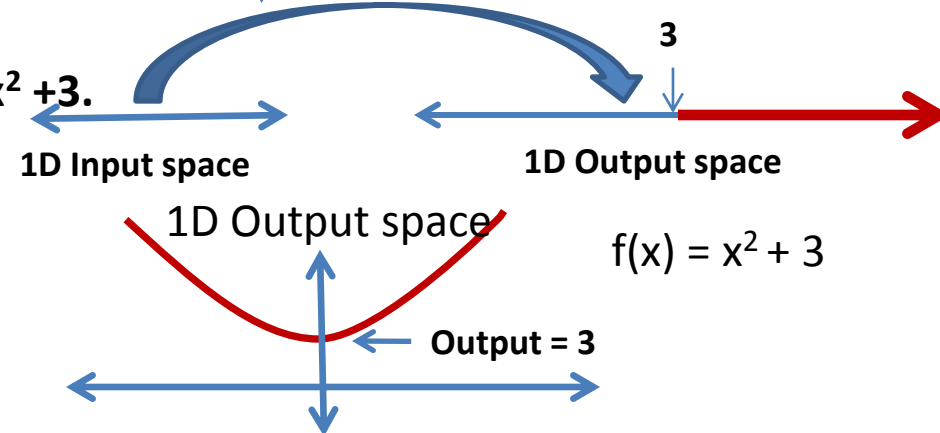


Now suppose I change the function to $f(x) = x^2 + 3$.

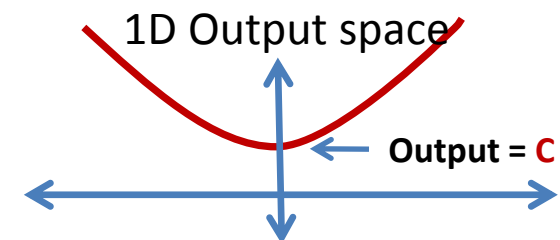
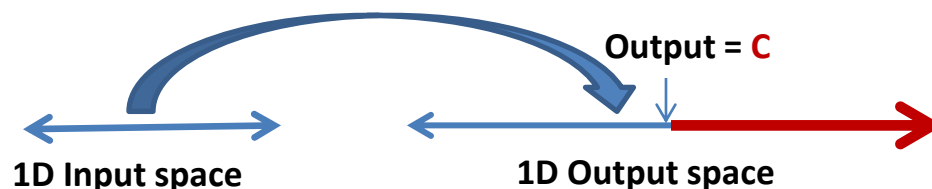
Still a mapping from $\mathbb{R}^1 \rightarrow \mathbb{R}^1$

If we plot output against input

then the graph is a different parabola



We can write something more general: $f(x) = x^2 + C$ (C is a constant)

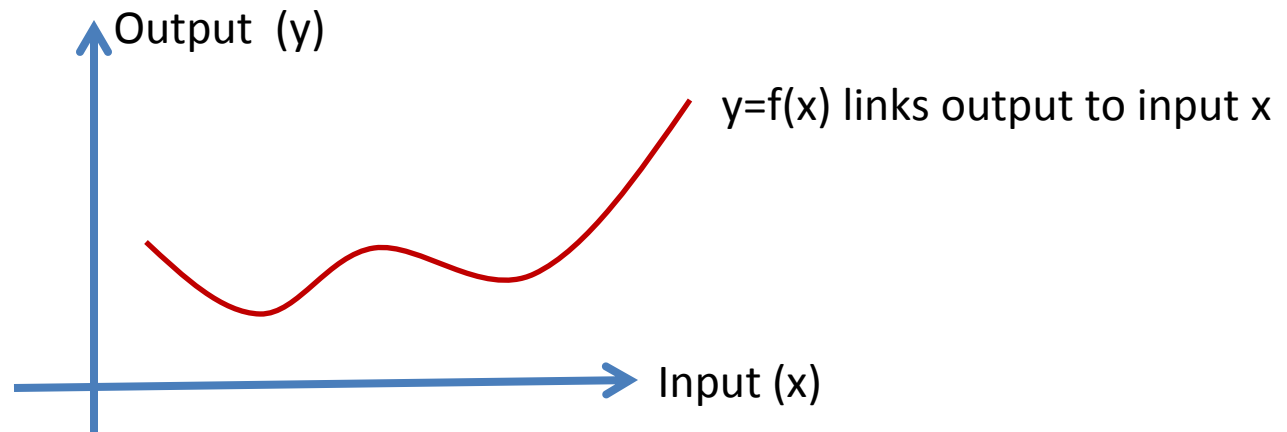


Section 14.1: Functions of Several Variables

The key (very important) point: **What do mean when we write $y = f(x)$?**

Realization: $f(x)$ is a function which maps $\mathbb{R}^1 \rightarrow \mathbb{R}^1$

By setting $y = f(x)$, then we linking a y variable to an input variable x



Again: Input (x) $\longrightarrow$ $f(x) = x^2$ $\longrightarrow$ Output (y)

I know, I know. You think I am crazy for making such a big deal out of this.
Hang on.

Section 14.1: Functions of Several Variables

Again: Input (x) $\longrightarrow$ $f(x) = x^2$ $\longrightarrow$ Output

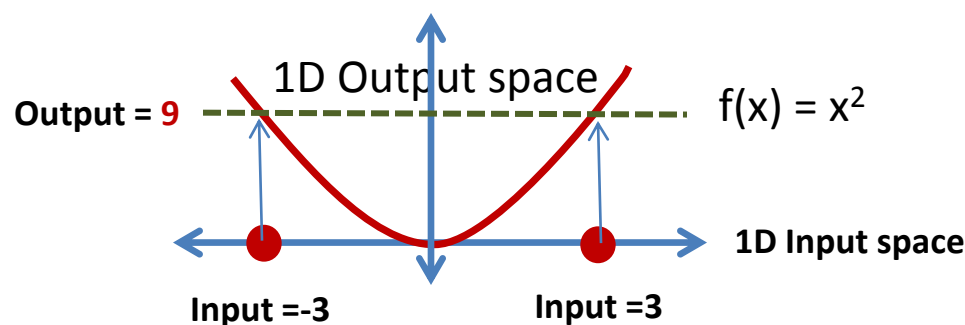
Now, suppose I ask:

what is the set of all inputs that get sent by $f(x) = x^2$ to the output 9 ?

Answer: the input values $x = -3$ and $x = 3$

Definition: Given $f(x)$, we call the “ k ” level set to be the set of all inputs that get sent to the output value of k .

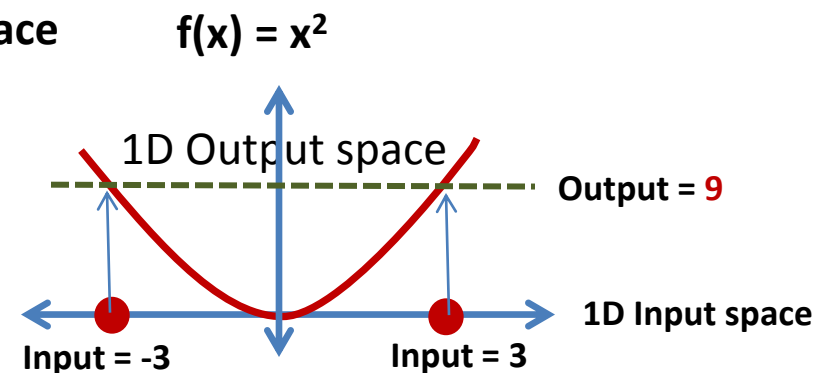
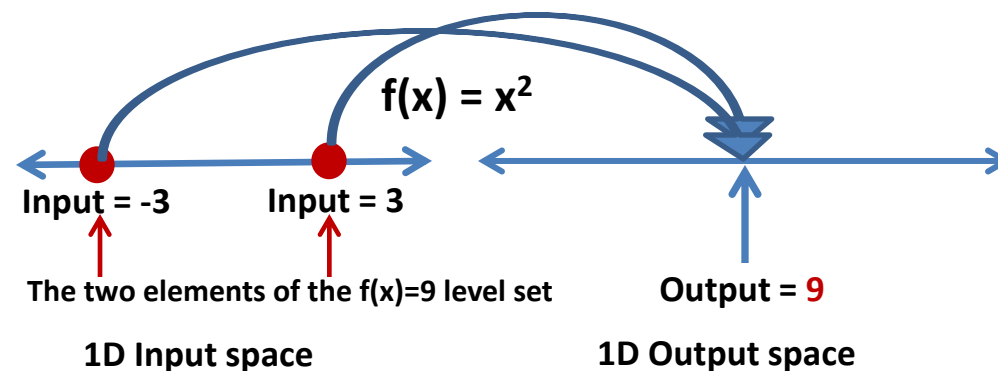
So, for the above example, the “ $k=9$ ” level set is the set of all inputs that get sent to output $y=9$, so the “ $k=9$ ” level set is the two inputs $x=-3$ and $x=3$



Section 14.1: Functions of Several Variables

Definition (again!): Given $f(x)$, we define the “ $f(x)=k$ ” level set to be the set of all inputs that get sent to the output value of k .

I want to stress that the level set lies in input space



The level set lies in ***input*** space

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Section 14.1: Functions of Several Variables

Example: $f(x) = \sin x$ What is the level set for $k=1$?

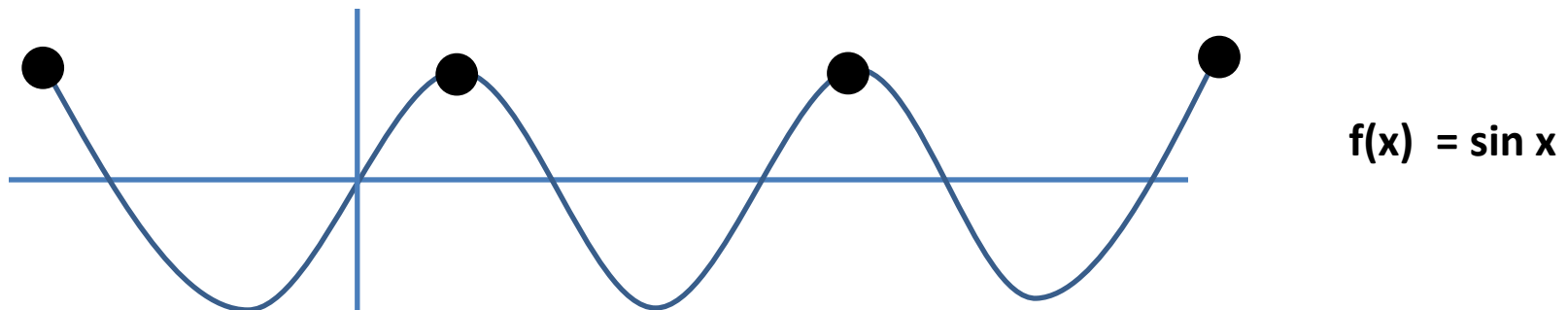
Answer: Step 1: We are trying to find the set of all inputs x such that the output $f(x)$ is $=1$

Step 2: So we need to find all x that satisfy $1 = f(x) = \sin x$

Step 3: So the answer is all input values x that are of the form

$$x = \frac{4n+1}{2} \pi \rightarrow \pi/2, 5\pi/2, \dots$$

where n is an integer (positive, negative, or zero)



Section 14.1: Functions of Several Variables

Now, let's go to a function of two variables and see what all this means:

Suppose $f(x,y) = x^2 + y^2$

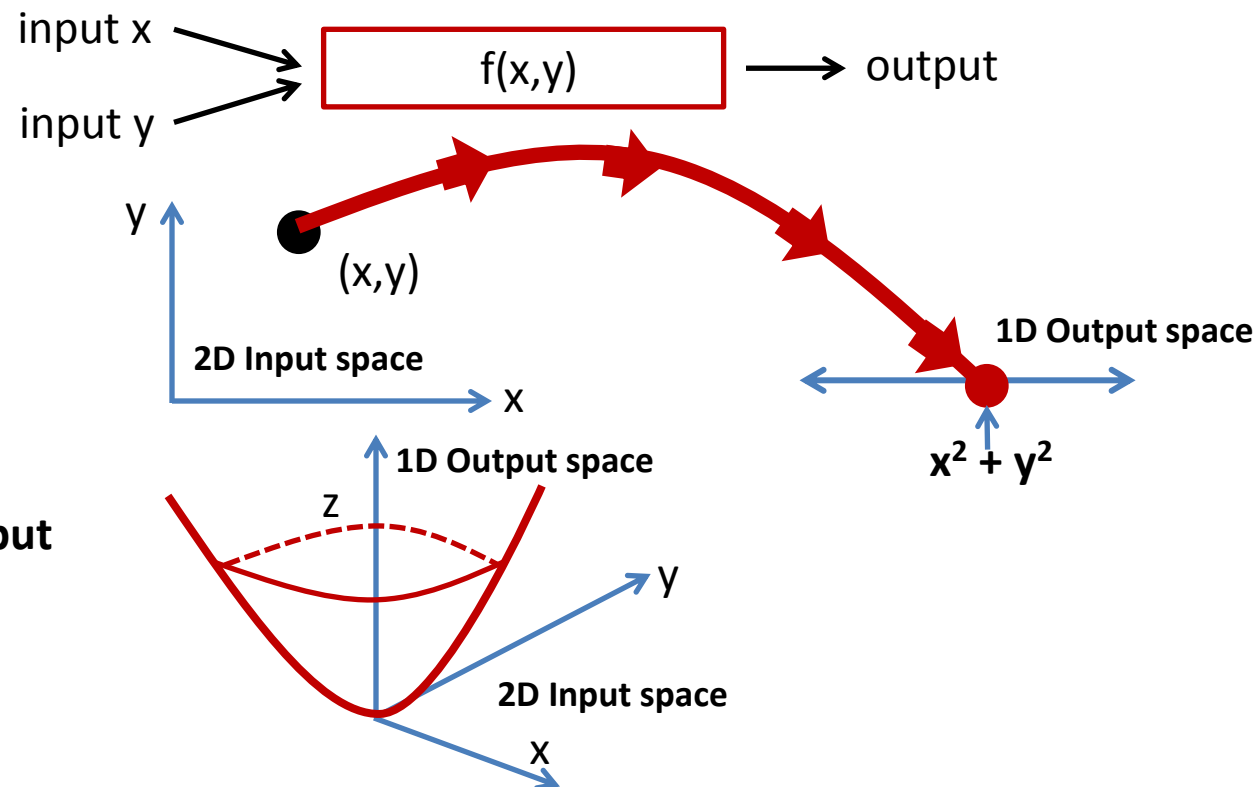
$$f(\vec{x}) \equiv f(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}^1$$

Then input space is 2D, and output space is 1D

We could choose to plot output space against input space:

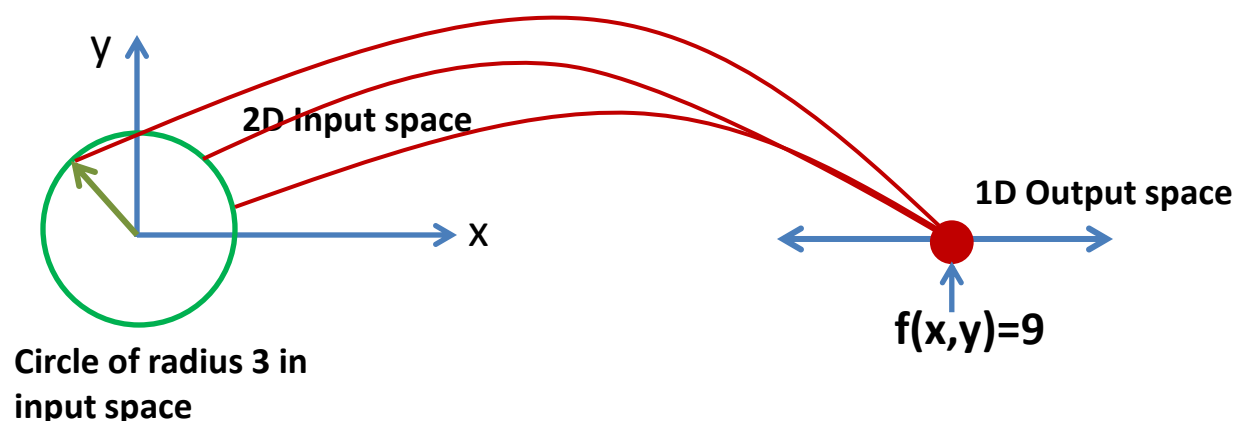
And now I could ask: what is the level set $f(x,y) = 9$?

Answer: it is the set of all input points that get sent to the output 9

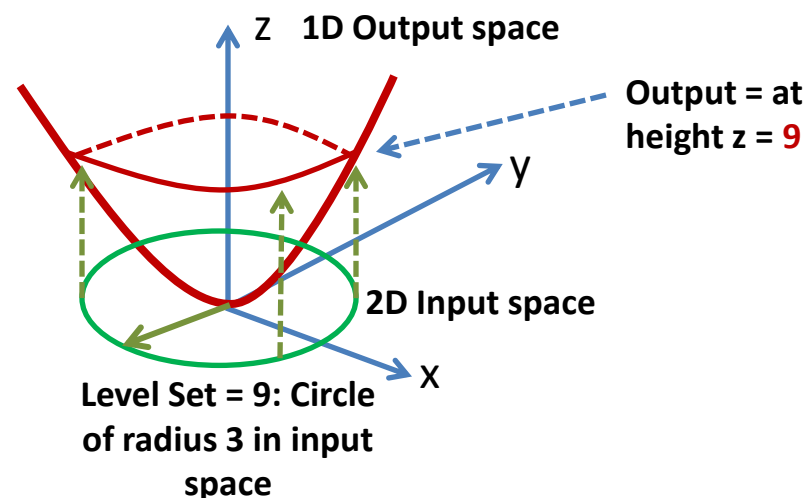


Section 14.1: Functions of Several Variables

$f(x,y) = x^2 + y^2$ What is the level set $f(x,y) = 9$?



We could choose to plot output space against input space:



Answer: it is the set of all input points that get sent to the output 9

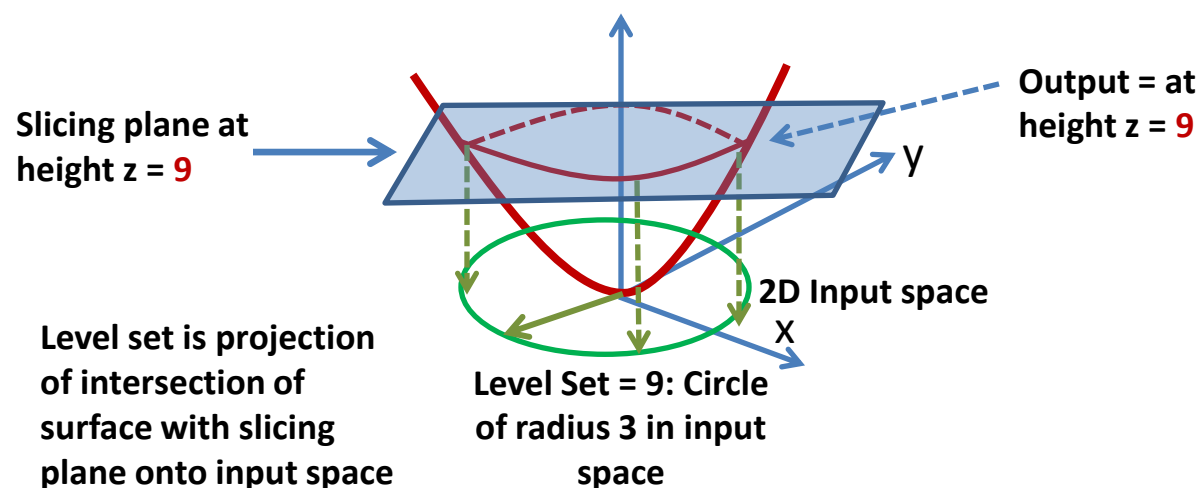
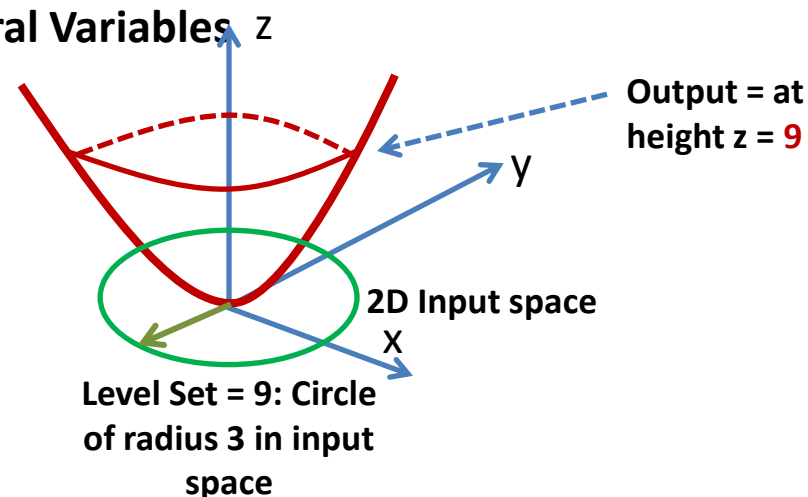
Section 14.1: Functions of Several Variables

$f(x,y) = x^2 + y^2$ What is the level set $f(x,y) = 9$?

Analytic answer:

it is the set of all input points that get sent to the output 9

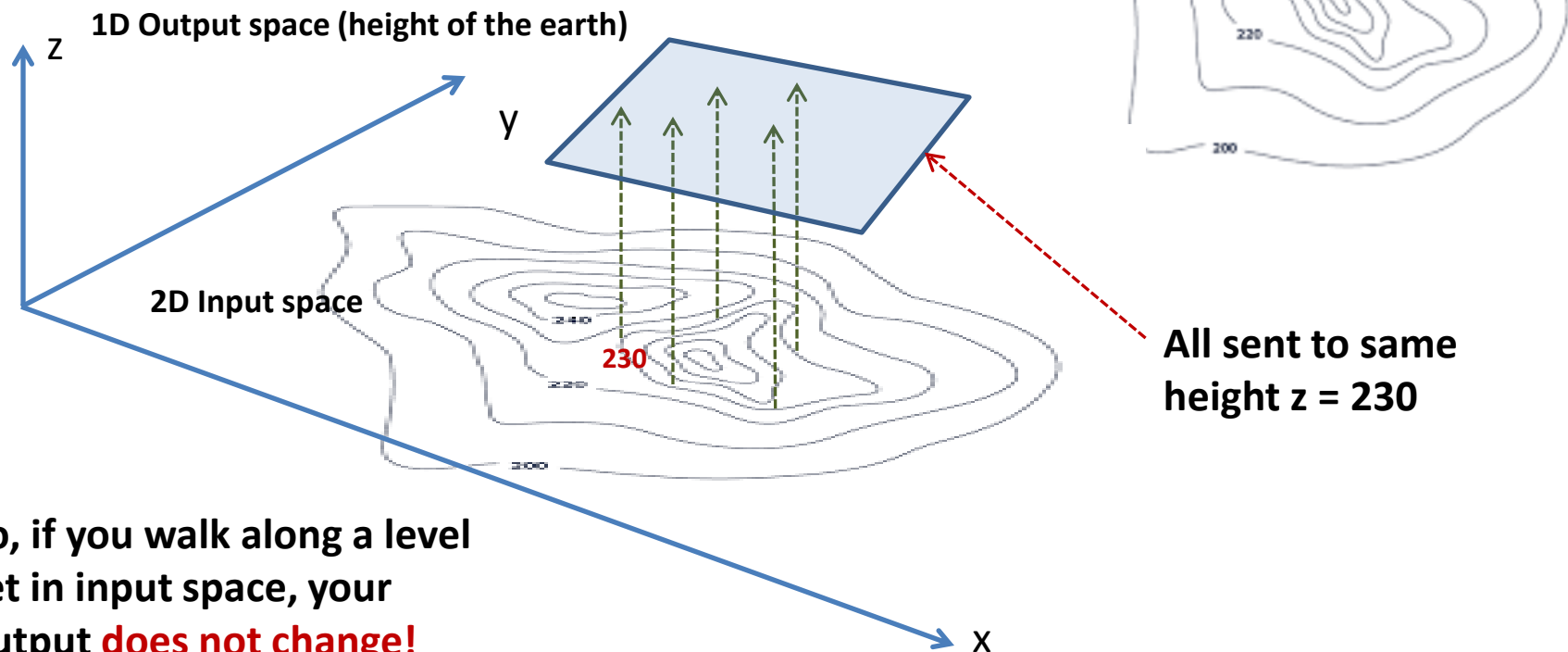
Geometric answer: if you slice the graph with a plane at height k , and project the result down to the input plane, the points you get are the k level set.



Section 14.1: Functions of Several Variables

And that's what a topographic map is!

A topographic gives you different k -level sets in the input space, showing what the height of the earth would be. Contour lines show points of the same elevation.



So, if you walk along a level set in input space, your output **does not change!**

Topo map from <https://datavizproject.com/data-type/topographic-map/>