About the final exam

(1) Exam date: Wed, Dec 17, 3-6PM.

(2) DSP students *must* send me an e-mail by Friday, Dec. 5th, 5PM, telling me that they have an accommodation letter specifying extra time.

(3) The exam will be *closed book*. You may use one sheet of paper (both sides) of your own notes.

Fundamental Theorem of Calculus

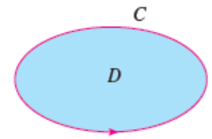
$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

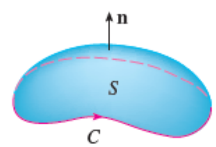
Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$$



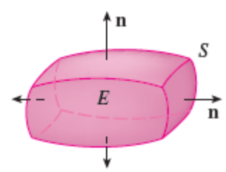
Stokes' Theorem

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$



Divergence Theorem

$$\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$



This chart in the book does a good job of summarizing the main "vector calculus" theorems, all but one of which we have proved.

Each one can be viewed as a higher dimensional version of the fundamental theory of calculus:

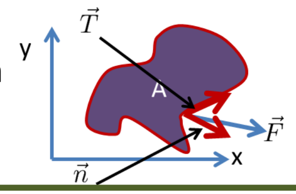
The integral of the "derivative" of a function is related to the value of that function on the boundary.

So now I can explain. Suppose you have:

A vector field
$\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$

+

A boundary C surrounding an interior surface S which could be either open or closed



You can't integrate a vector field on a boundary nor on an interior

But you can derive scalar functions from that vector field which then can be integrated

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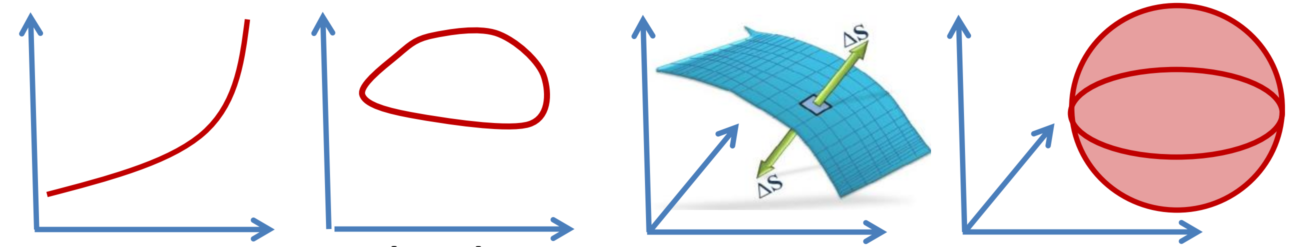
On the closed boundary C, you could either

- **Build a scalar function on the boundary consisting of normal components:** $\vec{F}(x,y,z) \cdot \vec{n}$ ← **this is a number at each point of the boundary C: So we can integrate all these numbers as we move around the boundary**
- **Build a scalar function on the boundary consisting of tangential components** $\vec{F}(x,y,z) \cdot \vec{\tau}$ ← **this is a number at each point of the boundary C: So we can integrate all these numbers as we move around the boundary**

In the interior S, you could

- **Build a scalar function in the closed interior by taking the divergence :** $\nabla \cdot \vec{F}$ ← **this is a number at each point of the interior: So we can integrate all these numbers as we move through the interior**
- **Build a scalar function in closed surface in 2D or an open surface in 3D by taking the curl :** $\nabla \times \vec{F}$ ← **Dotting this curl with the normal at each point of the interior gives a number: So we can integrate all these numbers as we move through the interior**

So given a function in a domain, we can evaluate that function on



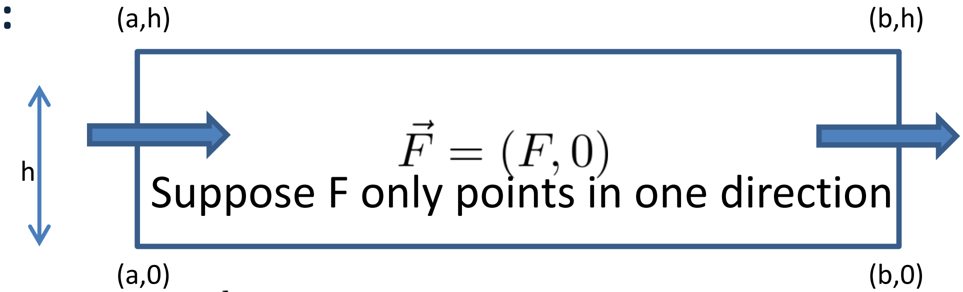
an open curve — **Fund. Theorm.Calc.**

a closed curve — **Green's and Divergence Thm.**

an open surface — **Stokes Thm.**

a closed surface — **Divergence Thm.**

Let's examine the divergence theorem again:



$$\int_S \vec{F} \cdot \vec{n} dS = \int_V \nabla \cdot \vec{F} dV$$

So $\nabla \cdot \vec{F} = \frac{dF}{dx}$

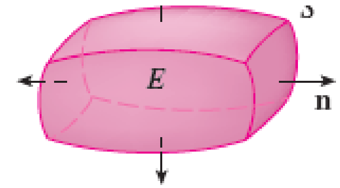
$$\text{So } \int_V \nabla \cdot \vec{F} dV = \int_V \frac{dF}{dx} dx dy = \int_0^h \int_a^b \frac{dF}{dx} dx dy = (h) \int_a^b \frac{dF}{dx} dx$$

$$\int_S \vec{F} \cdot \vec{n} dS = hF(b) - hF(a) = h\left[F(b) - F(a)\right]$$

$$\text{So } F(b) - F(a) = \int \frac{dF}{dx} dx$$

The fundamental theorem of calculus is a special case of the divergence theorem

Divergence Theorem $\iiint_E \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$



I am going to review this course by working backwards!

Then, you can understand why everything was needed in order to understand the above.

$$\int\int_S \vec{F} \cdot \vec{n}\, dS = \int\int\int_E \nabla \cdot \vec{F}\, dV$$

We needed to define how to do an integral over a surface, which meant understanding cross products and stretching of patches (16.6-16.7)

We needed to define normal and tangent vectors (12.1-12.3,)

We needed to define double and triple integration over regions (15.1-15.8)

We needed to define what was meant by a vector field: (16.1)

We needed to define the gradient operator and what is meant by a partial derivative (14.1-14.8)

We needed to define the dot product (12.1-12.3)

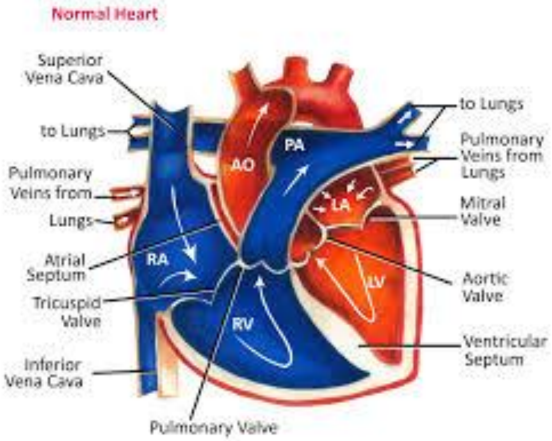
(and we also learned about curls, Jacobians, max/min, and Lagrange multipliers)!

Let's use all of this…..

The Navier-Stokes equations describe the flow of a fluid in complex settings (historical note—they pre-date Maxwell's equations for electromagnetism by roughly 20 years)

They describe

Weather and climate modeling: Every simulation that analyzes climate change uses these equations



Cardiac, pulmonary, and vascular modeling: Every simulation used to create heart valves, design stents, analyze aneurysms uses them

Ocean, waves, tsunami modeling. Every simulation uses these equations.

Forest fire modeling, combustion engines, plastic injection molding,…..

Okay—what are they?

Navier-Stokes equations:

Let $\vec{u}(x,y,z,t)$ **be the velocity of the fluid at (x,y,z) at time t**

Let $\rho(x,y,z,t)$ **be the density of the fluid at (x,y,z) at time t**

Let $P(x,y,z,t)$ **be the pressure of the fluid at (x,y,z) at time t**

Then (greatly simplifying), there are two fundamental equations

Eqn 1. $\rho\left(\mathbf{u_t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \mu\Delta u + \mathbf{s.t} + \mathbf{F}$

(mass)x(acceleration) = Force (pressure gradient + viscosity + surface tension + gravity)

(Momentum equation)

Eqn 2. $\nabla\cdot\mathbf{u} = \mathbf{0}$

Incompressibility requirement (you can't squeeze water)

Navier-Stokes Equations

$$\rho\,(\mathbf{u_t} + (\mathbf{u}\cdot\nabla)\mathbf{u}) = -\nabla p + \mu\Delta u + \mathbf{s.t} + \mathbf{F}$$ **(Momentum equation)**

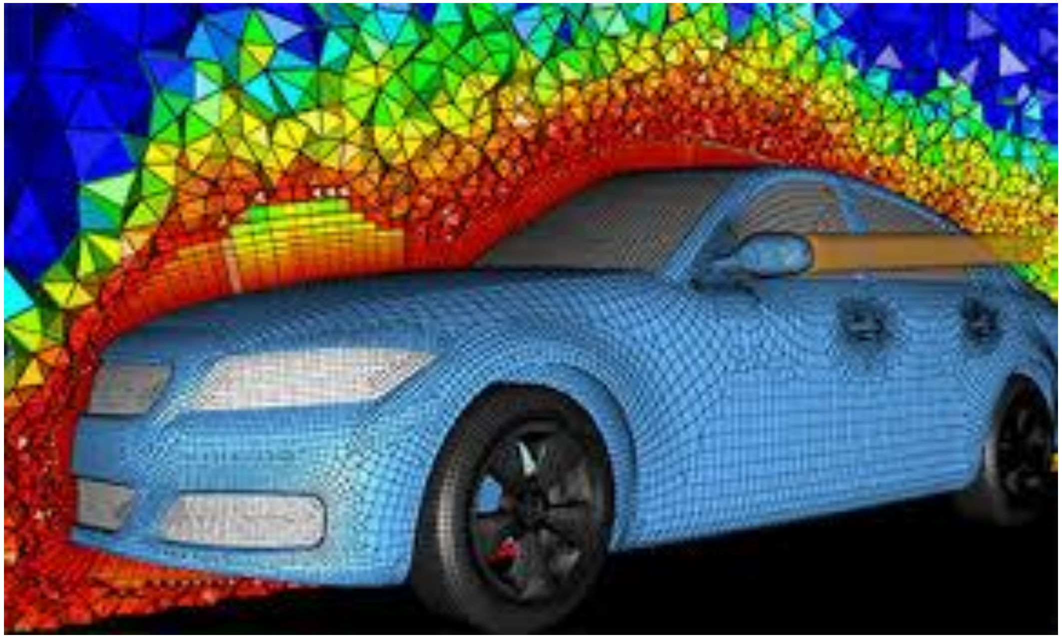
$$\nabla\cdot\mathbf{u} = \mathbf{0}$$ **Incompressibility requirement**

You now know what all these operators mean (gradient, dot product, divergence, partial derivatives)

How do you solve them? Answer: We approximate the equations

One approach:

(1) discretize space into a grid
(2) put all the variables (pressure, density, velocity) on the mesh
(3) approximate all the derivatives (in space and time) and update the values



Let's add one more complication:

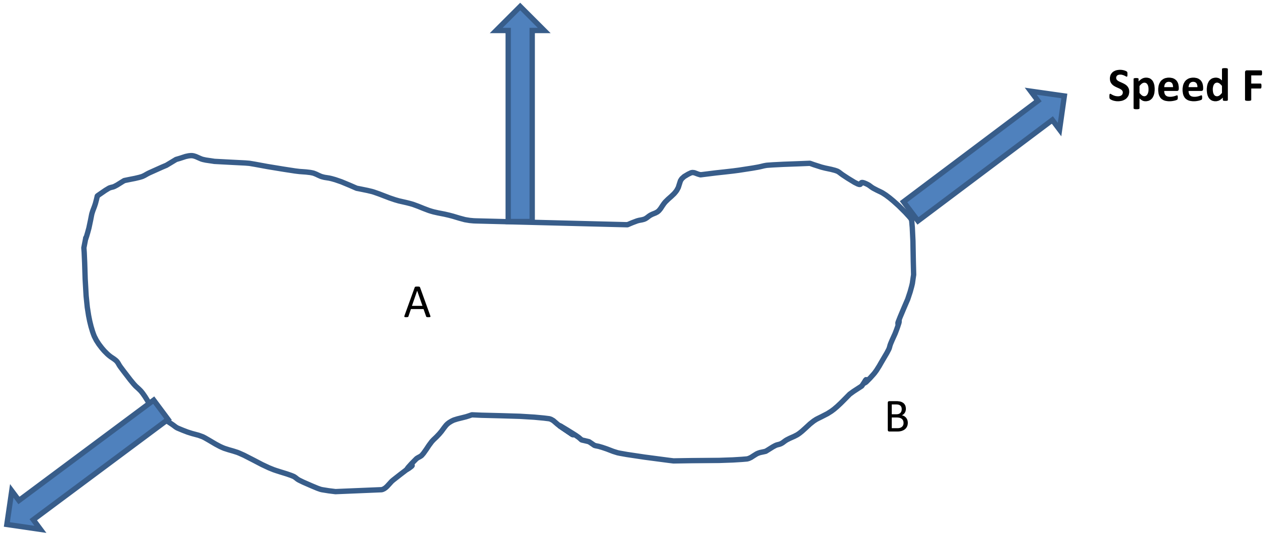
Suppose your domain contains an interface, separating different fluids:

Niagara Falls

This is an air-water interface (actually, lots of them!)

How can we build a method to approximate the interface when it moves?

Let's figure out a new way to track moving interfaces (using stuff you now know!)



Region A is moving into region B with speed F

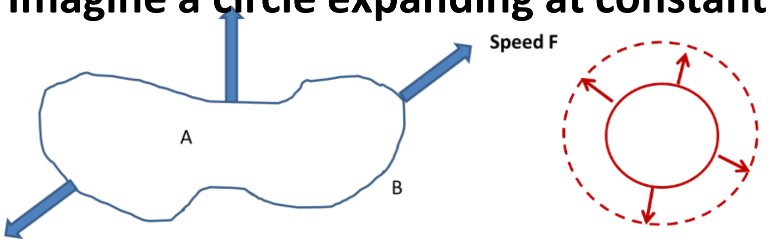
F can depend on everything (temperature, density, fluid flow,….

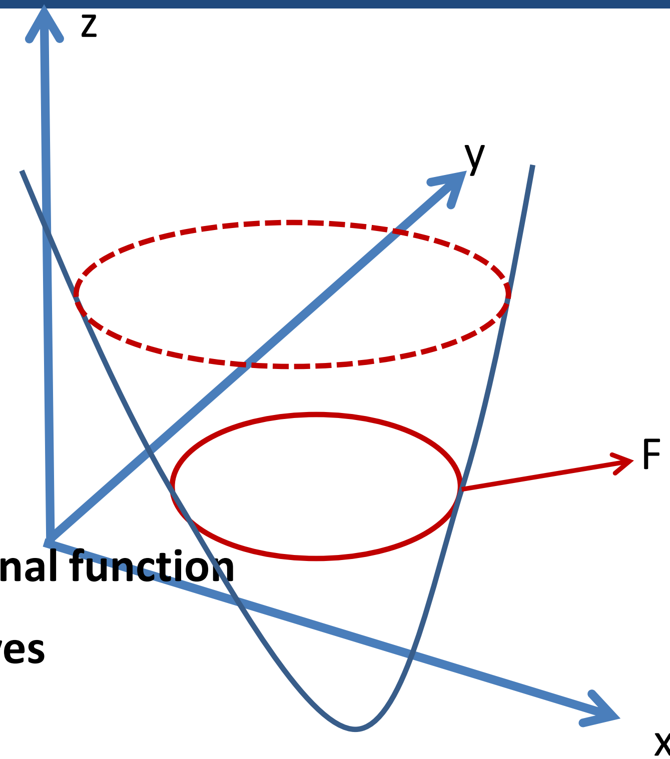
Examples: semiconductor modeling, blood flow, weather prediction, combustion, crystal growth, …….

How can we describe this moving "interface"?

A crash course in "level set methods"

Imagine a circle expanding at constant speed





Let's think about this in a different way.

Let d(x,y) be the distance from a point (x,y) to the interface

Let's build the function $z(x, y, t = 0) = \pm d(x, y)$

Then the interface is the zero level set of a higher dimensional function

Can we write an equation for this function z(x,y,t) as it moves so that the zero level set moves with speed F?

YES!

(1) Let x(t), y(t) be the trajectory of a bug on the interface, moving with speed F

(2) Then we want z(x(t),y(t),t)=0 for all time (since the bug stays on the zero level set)

(3) Take the derivative of both sides with respect to t: (use the chain rule)

$$\frac{d[z(x(t), y(t), t)]}{dt} = (z_x x_t + z_y y_t + d(z)/dt) = z_t + (x_t, y_t) \cdot \nabla z = 0$$

(4) And the speed F in the normal direction is just $F = (x_t, y_t) \cdot \vec{n} = (x_t, y_t) \cdot \frac{\nabla z}{|\nabla z|}$

(5) So our equation is $z_t + F|\nabla z| = 0$ **Works in any number of dimensions!**

So now we have a way of solving moving interface problems in any number of dimensions

$$z_t + F|\nabla z| = 0$$

(we just used partial derivatives, gradients, normal vector, chain rule, level sets, ….)

Let's use it….

The last homework assignment

(1) Make your own sample final exam----

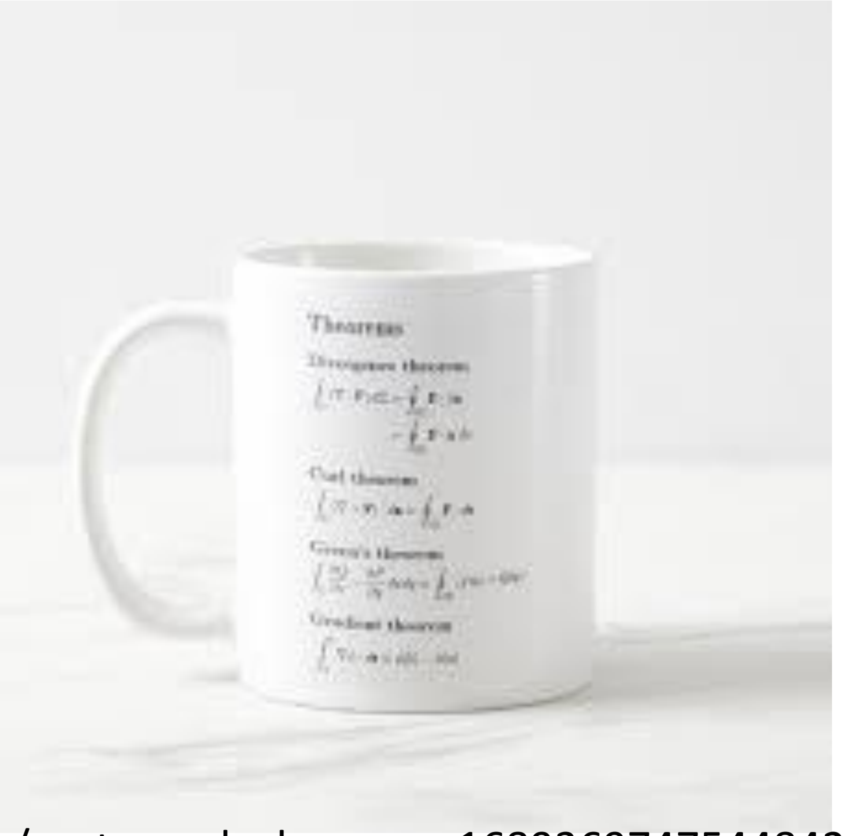
This is one of the best possible ways to prepare for the final.

(2) Make file cards, question on front, answers on back

Shuffle them, answer, shuffle, answer, repeat

And now you have earned the chance to go out and buy this mug:

Search for
“zazzle
multivariable mug”



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