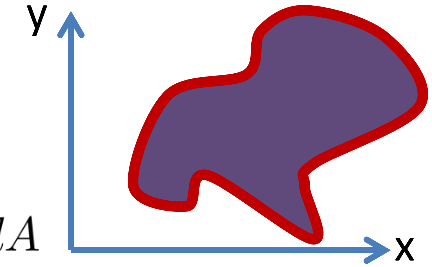
Theorems that relate integrals of derivatives of vector fields to boundary values

Green's theorem for 2D surface in the plane bounded by a 1D curve

$$\int_C \vec{F} \cdot d\vec{r} = \int \int_A \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dxdy$$

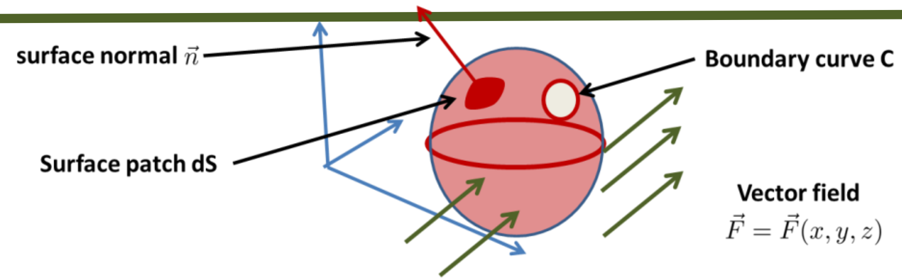
We can rewrite this:

Let $\vec{F} = (P(x,y), Q(x,y), 0) \Longrightarrow \int_C \vec{F} \cdot d\vec{r} = \int \int_A (\nabla \times \vec{F}) \cdot \vec{k} \; dA$



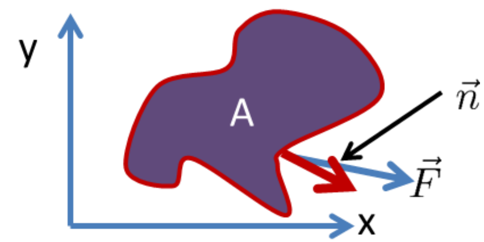
Green's theorem by 1D curve (Stokes theorem)

$$\int_C \vec{F} \cdot d\vec{r} = \int \int_S (\nabla \times \vec{F}) \cdot \vec{n} \; dS$$



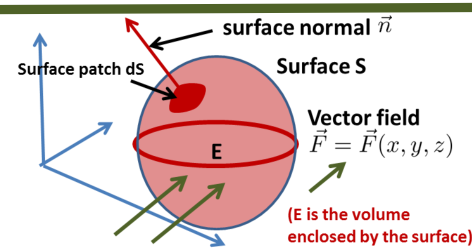
Divergence theorem for a 2D surface in the plane bounded by a 1D curve

$$\int_C \vec{F} \cdot \vec{n} ds = \int \int_A (\nabla \cdot \vec{F}) \; dA$$



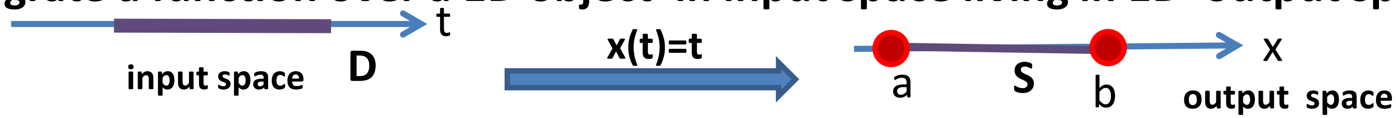
Divergence theorem for a 3D volume in 3D bounded by a 2D surface

$$\int \int_S \vec{F} \cdot \vec{n} dS = \int \int \int_E (\nabla \cdot \vec{F}) \; dE$$



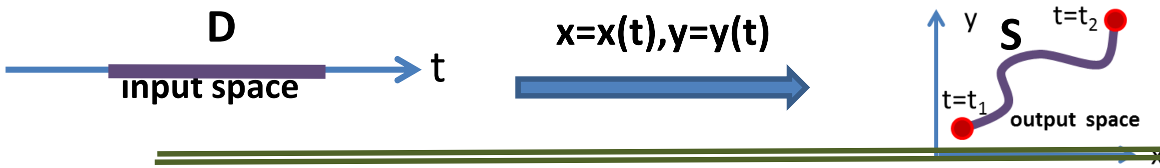
Integrating functions under a mapping

Integrate a function over a 1D object in input space living in 1D output space — LINE INTEGRAL



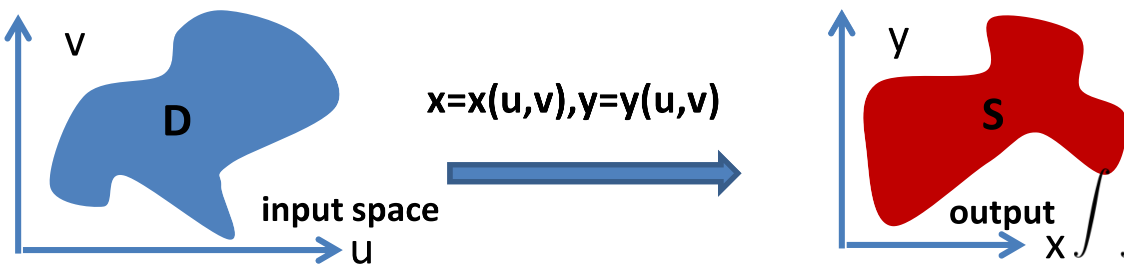
$$\int_a^b f(x)dx$$

Integrate a function over a 1D object in input space object living in 2D output space — LINE INTEGRAL



$$\int_{t_1}^{t_2} f(x(t),y(t))\sqrt{x_t^2+y_t^2}\,dt$$

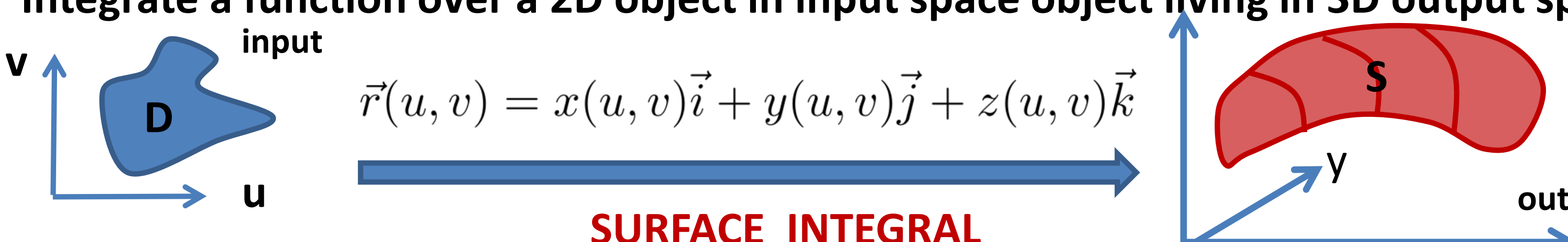
Integrate a function over a 2D object in input space object living in 2D output space — SURFACE INTEGRAL



$$\int\int_S f(x,y)dxdy = \int\int_D f(x(u,v),y(u,v))\left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right|dudv$$

Integrate a function over a 2D object in input space object living in 3D output space

$$\vec{r}(u,v)=x(u,v)\vec{i}+y(u,v)\vec{j}+z(u,v)\vec{k}$$

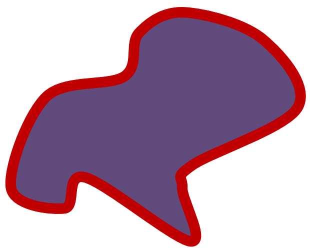


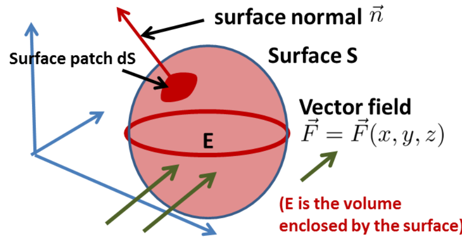
SURFACE INTEGRAL

$$\int\int_S f(x,y,z)dS = \int\int_D f(x(u,v),y(u,v),z(u,v)\ |\vec{r}_u\times\vec{r}_v|\ dudv$$

What do these two charts have to do with each other? Let me try and explain:
But first, I want to make sure that we are agree on what we are talking about:

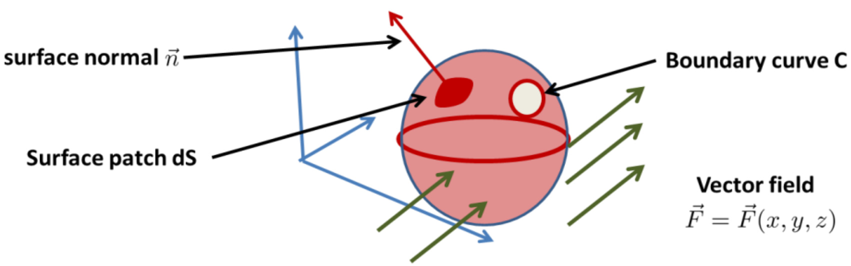
Def: A closed object separates a region into a clear inside and outside

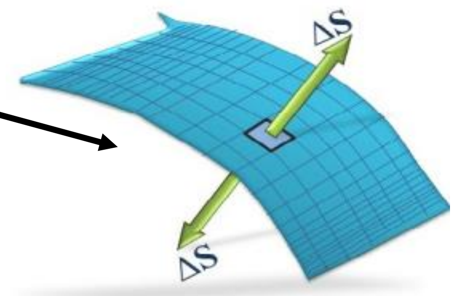




These are closed objects, because you can't get from one side of the contained region to the other without going through the boundary

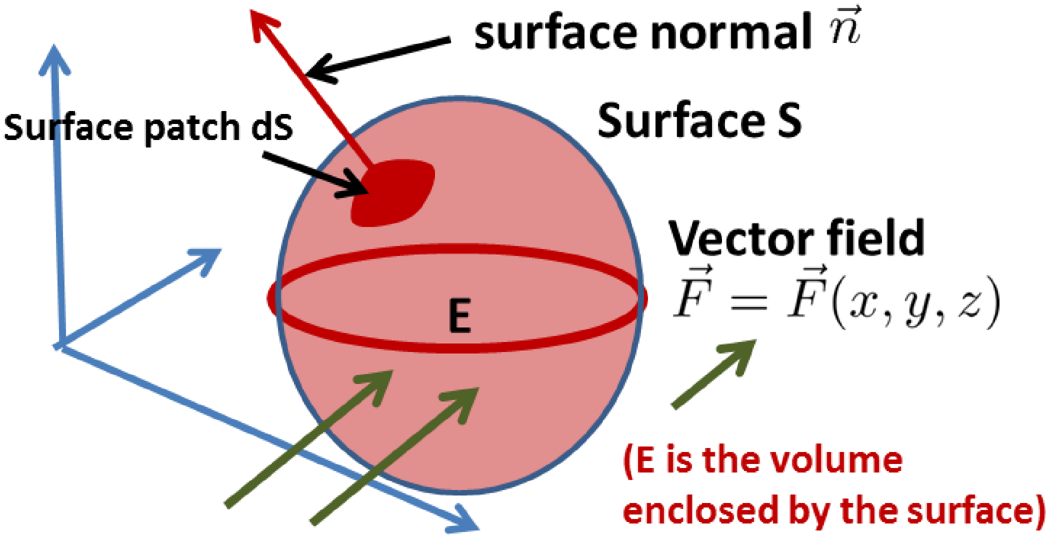
Def: An open object does not separate a region into a clear inside and outside



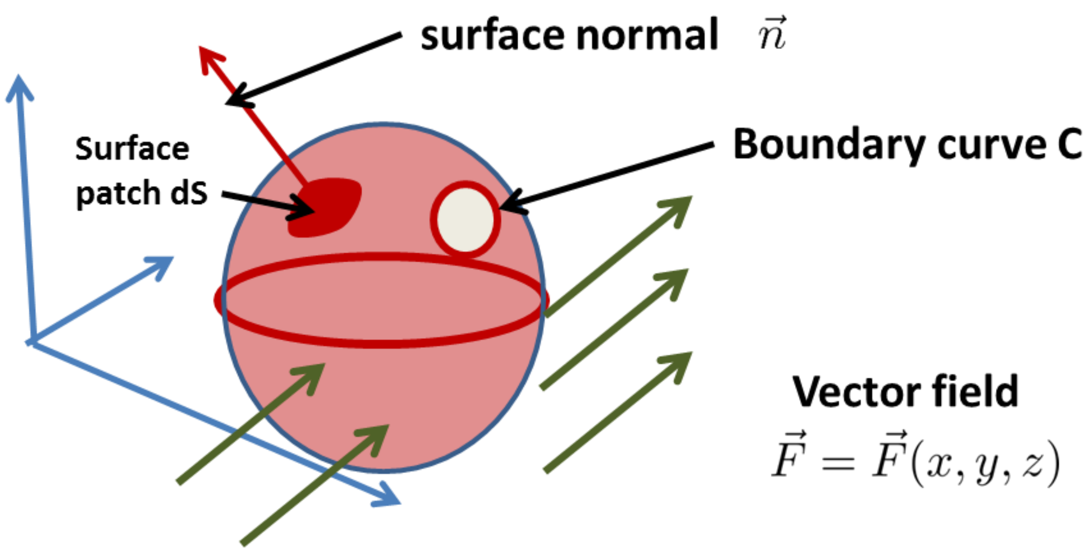


These are open objects, because you can get from one side to the other by going "around"

(right figure from https://studyingphysics.wordpress.com/2012/11/06/general-topics-of-physics/)



This is a closed 2D boundary surface lying in 3D and enclosing a 3D volume



This is a 1D boundary curve lying in 3D space enclosing an open 2D surface lying in 3D

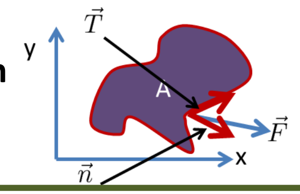
So now I can explain. Suppose you have:

A vector field

$$\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$$

+

A boundary C surrounding an interior surface S which could be either open or closed



You can't integrate a vector field on a boundary nor on an interior

But you can derive scalar functions from that vector field which then can be integrated

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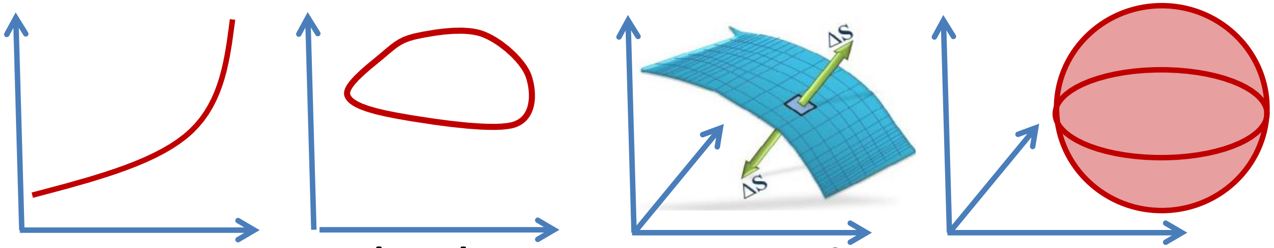
On the closed boundary C, you could either

- **Build a scalar function on the boundary consisting of normal components:** $\vec{F}(x,y,z) \cdot \vec{n}$ ← **this is a number at each point of the boundary C: So we can integrate all these numbers as we move around the boundary**
- **Build a scalar function on the boundary consisting of tangential components** $\vec{F}(x,y,z) \cdot \vec{\tau}$ ← **this is a number at each point of the boundary C: So we can integrate all these numbers as we move around the boundary**

In the interior S, you could

- **Build a scalar function in the closed interior by taking the divergence :** $\nabla \cdot \vec{F}$ ← **this is a number at each point of the interior: So we can integrate all these numbers as we move through the interior**
- **Build a scalar function in closed surface in 2D or an open surface in 3D by taking the curl :** $\nabla \times \vec{F}$ ← **Dotting this curl with the normal at each point of the interior gives a number: So we can integrate all these numbers as we move through the interior**

So given a function in a domain, we can evaluate that function on



an open curve — Fund. Theorm.Calc.

a closed curve — Green's and Divergence Thm.

an open surface — Stokes Thm.

a closed surface — Divergence Thm.

So, let's put all this together in a screwy way:

Consider the mapping from R to R²

$$x(t) = [2 + .1 * (\sin(16\pi t)] \cos(2\pi t)$$

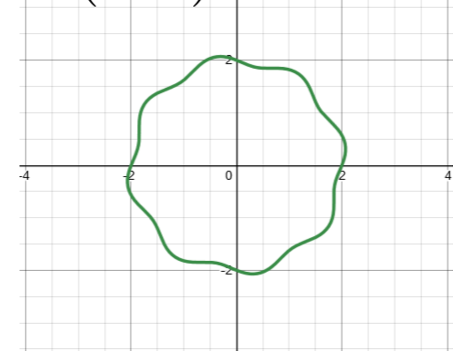
$$y(t) = [2 + .1 * (\sin(16\pi t)] \sin(2\pi t)$$

$$0 \le t \le 1$$

x(t),y(t)

L = line segment)

input space

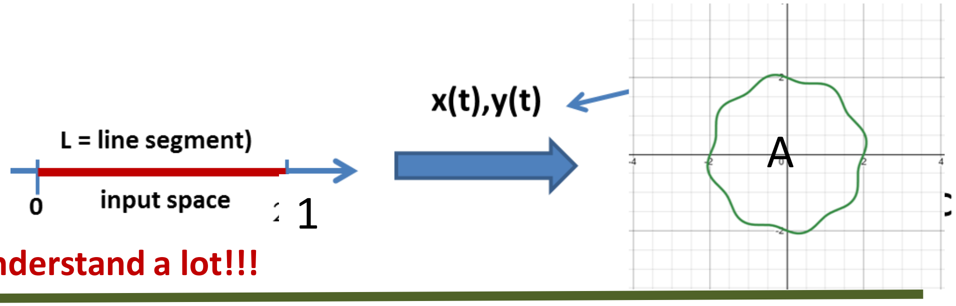


This is a parameterization of a curve in output space

$$x(t) = [2 + .1 * (\sin(16\pi t)] \cos(2\pi t)$$

$$y(t) = [2 + .1 * (\sin(16\pi t)] \sin(2\pi t)$$

Using this mapping, find a line integral over L in input space for the area of the region A on the right



If you understand this problem, you understand a lot!!!

We will do this problem by converting a vector integral over the interior in output space into a line integral in output space, and then convert that into a line integral in input space using the mapping from input space to output space

Let $F(x, y) = (P(x, y), Q(x, y)) = (.5x, .5y)$

Then Area = $\int\int_A [1]\, dA = \int\int_A \nabla \cdot \vec{F}\, dA = \int_C \vec{F} \cdot \vec{n}\, dC$

Tangent

$$x_t = ([.1 * 16\pi \cos(16\pi t)] \cos(2\pi t) + [2 + .1 \sin(16\pi t)] [-2\pi \sin(2\pi t)]$$

$$y_t = ([.1 * 16\pi \cos(16\pi t)] \sin(2\pi t) + [2 + .1 * \sin(16\pi t)] [2\pi \cos(2\pi t)]$$

So normal: $\vec{n} = \frac{(y_t, -x_t)}{[x_t^2 + y_t^2]^{\frac{1}{2}}}$

Stretch factor

So, $\int_C \frac{x * y_t - y * x_t}{2[x_t^2 + y_t^2]^{\frac{1}{2}}}\, dC = \int_L \frac{x * y_t - y * x_t}{2[x_t^2 + y_t^2]^{\frac{1}{2}}} \boxed{(x_t^2 + y_t^2)^{\frac{1}{2}}}\, dt = \frac{1}{2}\int_{t=0}^{t=1} [x * y_t - y * x_t]\, dt$

$$= \frac{1}{2}\int_{t=1}^{t=1} [[2 + .1 * (\sin(16\pi t)] \cos(2\pi t)] \;\; [(([.1 * 16\pi \cos(16\pi t)] \cos(2\pi t) + [2 + .1 * 16 \sin(16\pi t)] [-2\pi \sin(2\pi t)]]$$

$$+$$

$$[[2 + .1 * (\sin(16\pi t)] \sin(2\pi t)][(([.1 * 16\pi \cos(16\pi t)] \sin(2\pi t) + [2 + .1 * 16 \sin(16\pi t)] [2\pi \cos(2\pi t)]]\, dt$$

Okay –I now owe you "proofs" of the divergence theorem and Stokes' theorem

We start with the divergence theorem: want to prove that

$$\int\int_S \vec{F}\cdot\vec{n}ds = \int\int\int_E (\nabla\cdot\vec{F})\; dE \qquad ***$$

Step 1: Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ **Then:** $\nabla\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

right side of * is** $\int\int\int_E \nabla\cdot\vec{F}\,dE = \int\int\int_E \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right] dE = \boxed{\int\int\int_E \left[\frac{\partial P}{\partial x}\right] dE} + \ldots$

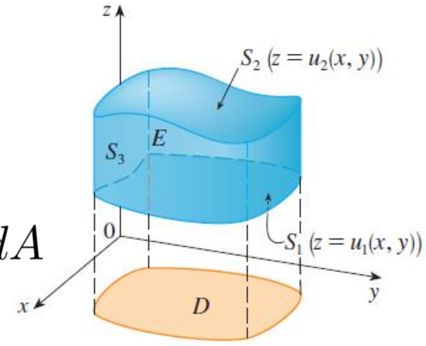
and the left side of * is** $\int\int_S \vec{F}\cdot\vec{n}ds = \int\int_S (P\vec{i} + Q\vec{j} + R\vec{k})\cdot\vec{n}dS$

$$= \boxed{\int\int_S (P\vec{i}\cdot\vec{n})dS} + \int\int_S (Q\vec{j}\cdot\vec{n})dS + \int\int_S (R\vec{k}\cdot\vec{n})dS$$

Step 2: So we can prove the divergence theorem if we show that

$$\boxed{\int\int_S (P\vec{i}\cdot\vec{n})dS = \int\int\int_E \left[\frac{\partial P}{\partial x}\right] dE} \quad \boxed{\int\int_S (Q\vec{j}\cdot\vec{n})dS = \int\int\int_E \left[\frac{\partial Q}{\partial y}\right] dE} \quad \boxed{\int\int_S (R\vec{k}\cdot\vec{n})dS = \int\int\int_E \left[\frac{\partial R}{\partial z}\right] dE}$$

Step 3: Let's prove the last one. Assume a "type 1" region:

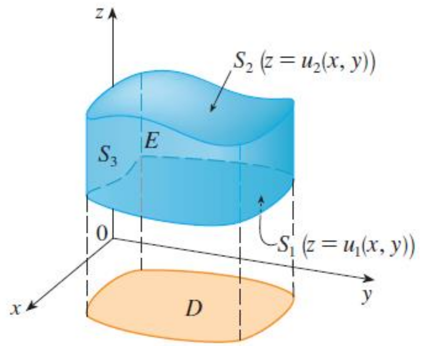


$$\int\int\int_E \left[\frac{\partial R}{\partial z}\right] dE = \int\int_D \left[\int_{u_1(x,y)}^{u_2(x,y)} \frac{\partial R}{\partial z}(x,y,z)dz\right] dA$$

Use fundamental theor. of calc. $= \int\int_D \left[R(x,y,u_2(x,y)) - R(x,y,u_1(x,y)\right] dA$

= an integral over the top surface – an integral over the bottom surface

We are trying to show that $\int\int_S (R\vec{k}\cdot\vec{n})dS = \int\int\int_E \left[\frac{\partial R}{\partial z}\right] dE$ ****



We just tackled the right side of ****

$$\int\int\int_E \left[\frac{\partial R}{\partial z}\right] dE = \int\int_D \left[\int_{u_1(x,y)}^{u_2(x,y)} \frac{\partial R}{\partial z}(x,y,z)dz\right] dA$$

$$= \int\int_D [R(x,y,u_2(x,y)) - R(x,y,u_1(x,y)] \, dA$$

= an integral over the top surface – an integral over the bottom surface

Let's now tackle the left side of ****

$$\int\int_{top} (R\vec{k}\cdot\vec{n})dS = \int\int_D R(x,y,u_2(x,y))dA$$

dA is a element of the projection of surface onto plane

$$\int\int_{bottom} (R\vec{k}\cdot\vec{n})dS = \int\int_D -R(x,y,u_1(x,y))dA$$

this has a minus sign because the normal points down

So, $\int\int_S (R\vec{k}\cdot\vec{n})dS = \int\int_D [R(x,y,u_2(x,y)) - R(x,y,u_1(x,y))] \, dA$

Which is exactly what we had from the right side

The same type of argument shows that

$$\int\int_S (P\vec{i}\cdot\vec{n})dS = \int\int\int_E \left[\frac{\partial P}{\partial x}\right] dE \qquad \int\int_S (Q\vec{j}\cdot\vec{n})dS = \int\int\int_E \left[\frac{\partial Q}{\partial y}\right] dE$$

So, we've proved the divergence theorem $\int\int_S \vec{F}\cdot\vec{n}ds = \int\int\int_E (\nabla\cdot\vec{F}) \, dE$

The proof of Stokes' theorem has the same flavor: I leave it you to read it in the book…

In the two lectures, I will review the course, and talk about the mechanics of the final exam. And then tell you about state-of-the-art research that needs all this stuff.