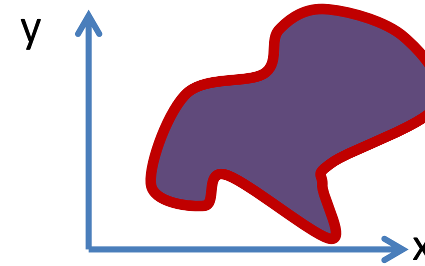
Stokes Theorem (16.8) and the Divergence Theorem (16.9)



Last Time: Green's theorem for a region in the plane

$$\int_C \vec{F} \cdot d\vec{r} = \int\int_A \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dxdy$$

And we had a fancy way to write Green's theorem: Suppose $\vec{F} = (P(x,y), Q(x,y), 0)$

(no z component)

Then we have that $\nabla \times \vec{F} = (0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})$

So now we can write Green's theorem as

$$\int_C \vec{F} \cdot d\vec{r} = \int\int_A (\nabla \times \vec{F}) \cdot \vec{k}\ dA$$

So, we can think of Green's theorem as the following:

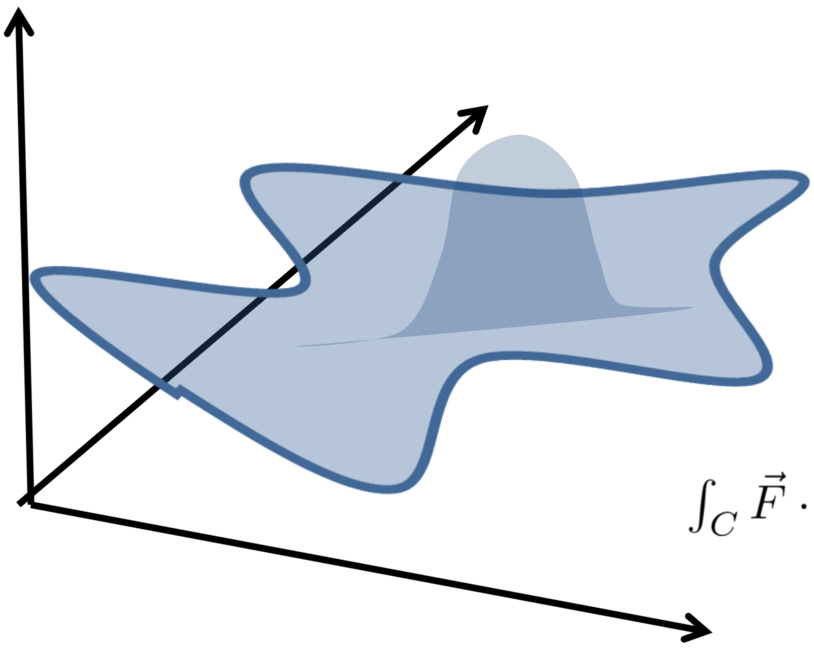
Collecting the tangential component of a vector field around a closed curve	**=**	**The total flux of the vector field $\nabla \times \vec{F}$ through the surface enclosed by that boundary**

$$\int_C \vec{F} \cdot d\vec{r} = \int\int_A (\nabla \times \vec{F}) \cdot \vec{k} \; dA$$

Collecting the tangential component of a vector field around a closed curve = **The total flux of the curl of a vector field through the surface enclosed by that boundary**

Stokes' Theorem

The same thing is true in 3D!!!



(this is supposed to be a surface in 3D!).

$$\int_C \vec{F} \cdot d\vec{r} = \int\int_S (\nabla \times \vec{F}) \cdot \; d\vec{S}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int \int_A (\nabla \times \vec{F}) \cdot \vec{k} \; dA$$

Collecting the tangential component of a vector field around a closed curve = **The total flux of the curl of the vector field through the surface enclosed by that boundary**

Which immediately leads to a remarkable statement:

$\vec{F} = \vec{F}(x, y, z)$

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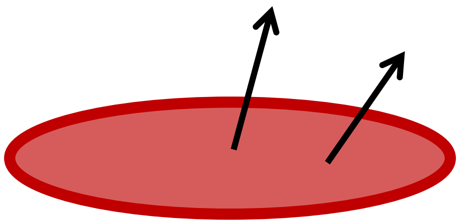

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Collecting the tangential component of a vector field around a closed curve

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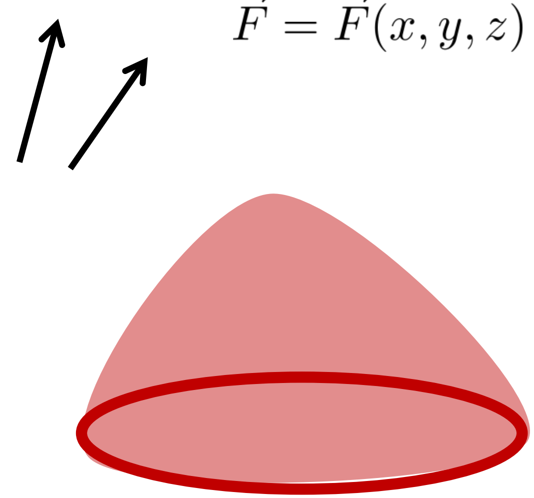
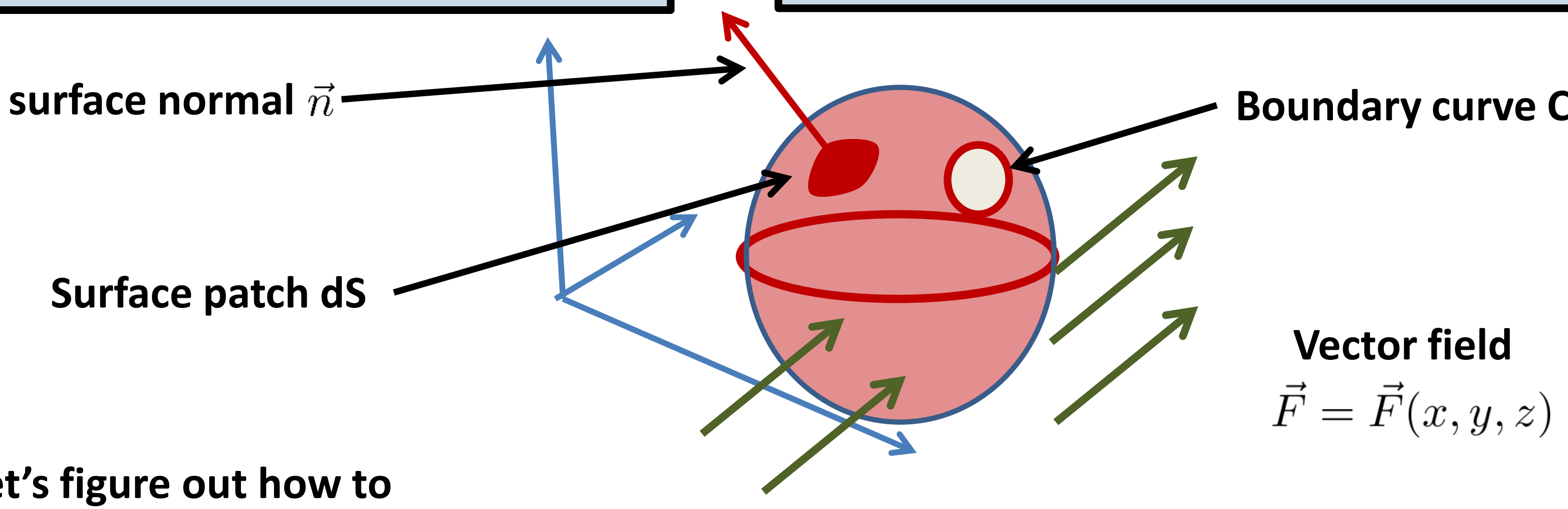


The total flux of the curl of a vector field through the surface enclosed by that boundary

These have to be the same!!!

The total flux of the curl of a vector field through the surface enclosed by that boundary

Collecting the tangential component of a vector field around a closed curve = **The total flux of the curl of the vector field through the surface enclosed by that boundary**



Let's figure out how to write this theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \int \int_S (\nabla \times \vec{F}) \cdot \vec{n} \; dS$$

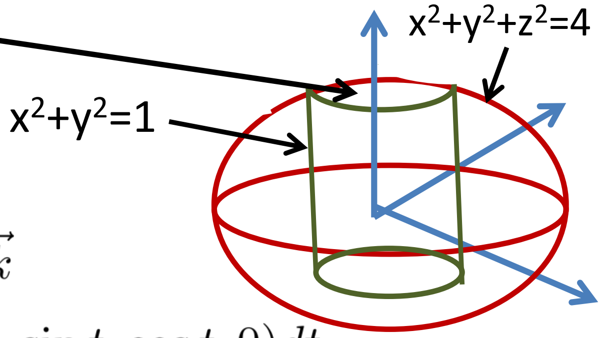
Let's use it!

$\int_C \vec{F} \cdot d\vec{r} = \int \int_A (\nabla \times \vec{F}) \cdot \vec{n} \ dS$ **Example: Compute** $\int \int_S curl \vec{F} \cdot d\vec{S} = \int \int_S \nabla \times \vec{F} \cdot d\vec{S}$

where $\vec{F}(x,y,z) = xz\vec{i} + yz\vec{j} + xy\vec{k}$

and S is the part of the sphere $x^2+y^2+z^2=4$ inside the cylinder $x^2 + y^2 =1$ and above the xy plane

Step 1: Make a drawing: the region is the "polar ice cap"



Step 2: Find the boundary curve

$x^2 + y^2 + z^2 = 4$ and $x^2+y^2=1$ so $1 + z^2 =4$

So the boundary curve is $x^2+y^2=1$ and $z = \sqrt{3}$

Step 3: Parameterize boundary $\vec{r}(t) = \cos t \ \vec{i} + \sin t \ \vec{j} + \sqrt{3} \ \vec{k}$

Step 4: Find parameterized stretch factor $d\vec{r}(t) = \vec{r}\,'(t)dt = (-\sin t, \cos t, 0)dt$

Step 5: Evaluate the vector field in the parameterized coordinates

$$\vec{F}(\vec{r}(t)) = xz \ \vec{i} + yz \ \vec{j} + xy \ \vec{k} = \sqrt{3}\cos t \ \vec{i} \ + \sqrt{3}\sin t \ \vec{j} \ + \ \cos t \sin t \ \vec{k}$$

Step 6: Set up the integral

$$\int \int_A (\nabla \times \vec{F}) \cdot \vec{n} \ dS = \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{r}\,' dt = \int_0^{2\pi} \left(-\sqrt{3}\cos t \sin t + \ \sqrt{3} \sin t \cos t\right) dt$$

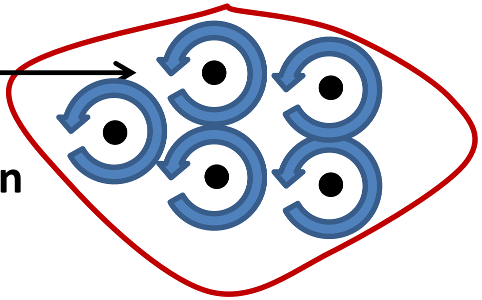
Step 7: Do the integral!

$$\int_0^{2\pi} (0) \ dt = 0$$

$$\int_C \vec{F} \cdot d\vec{r} = \int \int_A (\nabla \times \vec{F}) \cdot \vec{n} \; dS$$

A partial proof is in the book—let me try and give you a "geometric feel" for why it's true.

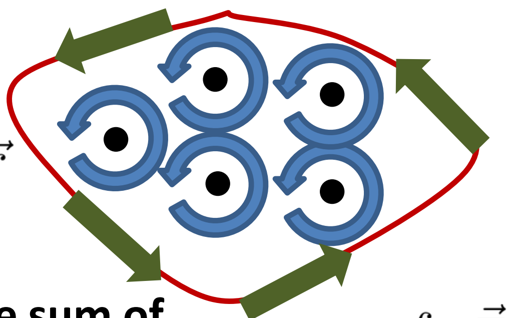
$(\nabla \times \vec{F})$ **is the "twist" at a point in the domain**



So $\int \int_A (\nabla \times \vec{F}) \cdot \vec{n} \; dS$ **is adding up all the twists in the domain**

All those twists together will fight with each other, causing the entire object to rotate

$\int_C \vec{F} \cdot d\vec{r}$



CAUTION: THIS IS *NOT* A PROOF!!!

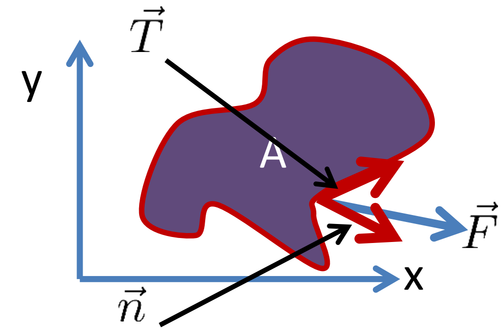
And the total rotation is just the sum of the vector field acting on the boundary

$$\int_C \vec{F} \cdot d\vec{r}$$

So: $\int_C \vec{F} \cdot d\vec{r} = \int \int_A (\nabla \times \vec{F}) \cdot \vec{n} \; dS$

And there was nothing about that argument that required the surface or the curve to be flat!

What else can we do??? One more amazing theorem!



So now we've found that collecting the tangential component of the vector field F yields: $\int_C \vec{F} \cdot d\vec{r} = \int \int_A (\nabla \times \vec{F}) \cdot \vec{n} \; dS$

And neither the enclosed domain nor the bounding curve need to be flat

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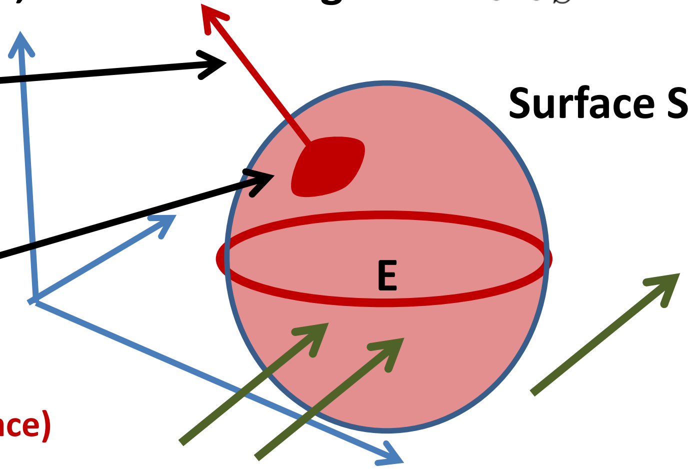
What happens if we try to collect the normal components of the vector field?

In 2D we have that $\int_C \vec{F} \cdot \vec{n} ds = \int \int_A (\nabla \cdot \vec{F}) \; dA$

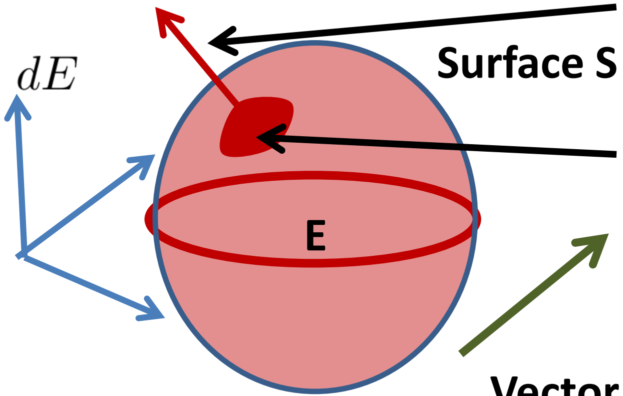
Is this theorem true in 3D? That is, is the following true? $\int \int_S \vec{F} \cdot \vec{n} ds = \int \int \int_E (\nabla \cdot \vec{F}) \; dE$



(E is the volume enclosed by the surface)

Vector field
$\vec{F} = \vec{F}(x, y, z)$

Yes!!! $\int \int_S \vec{F} \cdot \vec{n} dS = \int \int \int_E (\nabla \cdot \vec{F})\ dE$



surface normal $\vec{n}$

Surface S

Surface patch dS

(E is the volume enclosed by the surface)

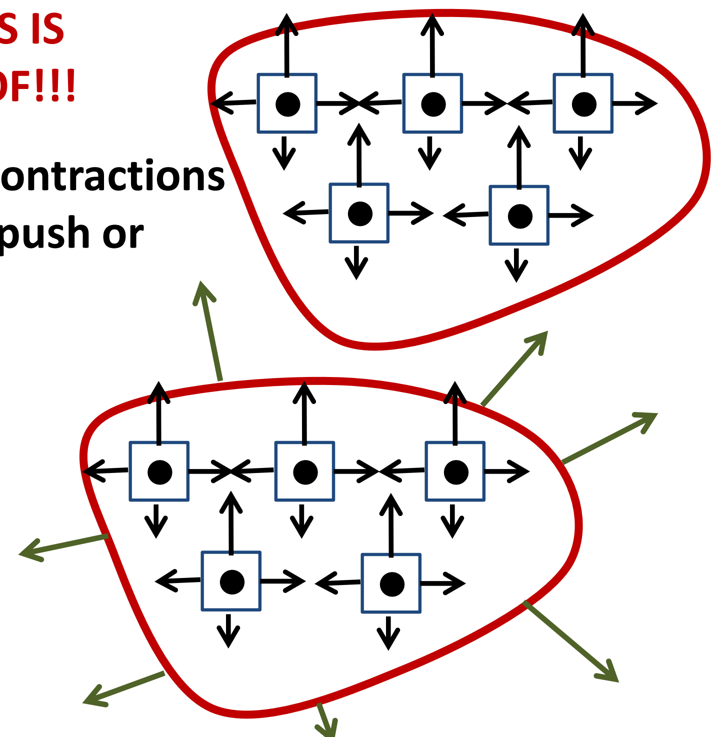
Again, let me give you a geometric feel as to why this is true….

Vector field $\vec{F} = \vec{F}(x, y, z)$

$\nabla \cdot \vec{F}$ **= Amount of expansion or contraction at a point**

CAUTION: THIS IS *NOT* A PROOF!!!

So $\int \int \int_E (\nabla \cdot \vec{F})\ dE$ **is adding up all the expansions/contractions in the domain, and letting them push or retract against each other**

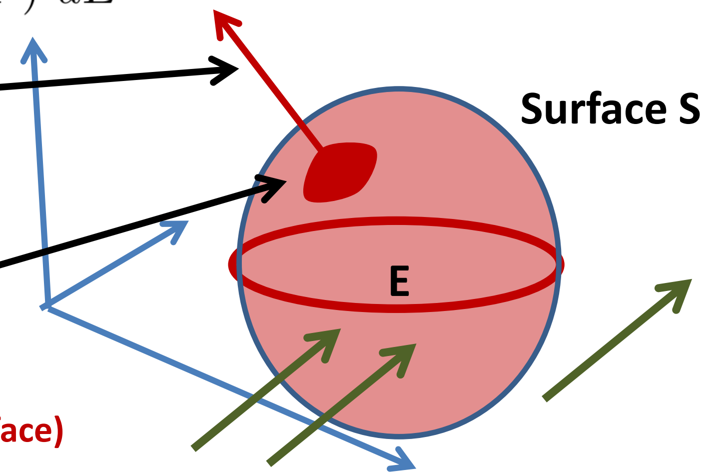


and when these all push against each other, the net effect is felt as a normal push on the boundary…

And there was nothing about that argument that required the surface or the curve to be in 2D!

$$\int \int_S \vec{F} \cdot \vec{n} dS = \int \int \int_E (\nabla \cdot \vec{F})\ dE$$

$$\int \int_S \vec{F} \cdot \vec{n} dS = \int \int \int_E (\nabla \cdot \vec{F}) \ dE$$



(E is the volume enclosed by the surface)

Let's use it

Example: Find $\int \int_S \vec{F} \cdot \vec{n} dS$ **where** $\vec{F} = z\vec{i} + y\vec{j} + x\vec{k}$ **and S is the sphere $x^2 + y^2 + z^2 = 1$**

Step 1: $\int \int_S \vec{F} \cdot \vec{n} dS = \int \int \int_E (\nabla \cdot \vec{F}) \ dE$

Step 2: Calculate $\nabla \cdot \vec{F} = \left(\frac{\partial z}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial x}{\partial z} \right) = 1$

Step 3: Set up and do the integral $= \int \int \int_E (1) \, dE$

=which is the integral of "1" over the unit sphere= volume of sphere = (4/3)π