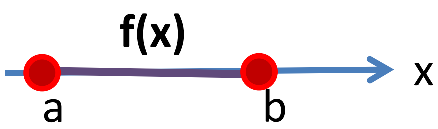
16.5 Last time:

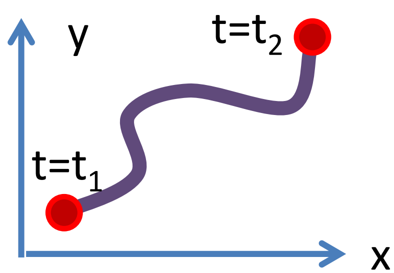
So far, we have established three theorems that relate the integral of the derivative of a function over a region to the values of the function on the boundary:

Boundary is a 0D object



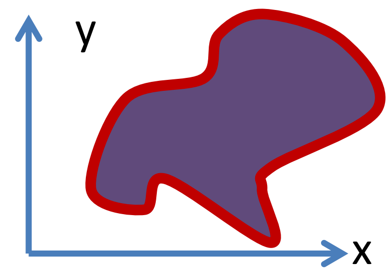
$$\int_a^b \frac{df}{dx} dx = f(b) - f(a)$$

Boundary is a 0D object



$$\int_{t_1}^{t_2} \nabla f \cdot d\vec{r} = f[\vec{r}(t_2)] - f[\vec{r}(t_1)]$$

Boundary is a 1D domain



$$\int_C \vec{F} \cdot d\vec{r} = \int \int_A \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dx dy$$

Suppose $\vec{F}$ is a vector field in R³ $\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$

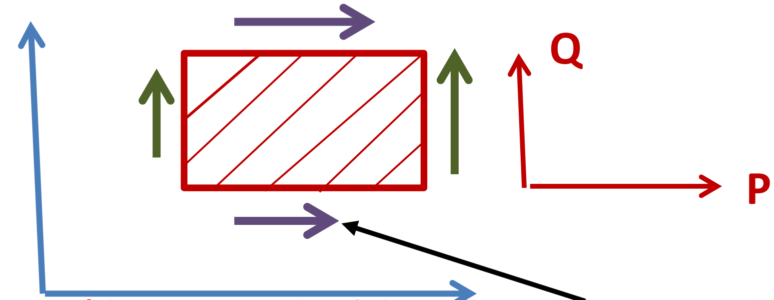
(at every point (x,y,z) in input space, it assigns a three dimensional vector)

We define the curl of $\vec{F}$ to be the vector $\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$

Let's look at a simpler vector field: $\vec{F} = (P, Q, R) = (P(x,y), Q(x,y), 0)$ **(the z component is zero)**

Then: $\nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \vec{k} = \left(0, 0, \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right)$

Let's draw a box in the xy plane



(the magnitude of these vectors corresponds to how strong the vector is)

$\frac{\partial P}{\partial y} =$ **the twisting of the box due to P**

If P is bigger on top than the bottom, then it twists clockwise 

If P is smaller on top than the bottom, then it twists counter clockwise 

$\frac{\partial Q}{\partial x} =$ **the twisting of the box due to Q**

If Q is bigger on left than the right, then it twists clockwise 

If Q is smaller on left than the right, then it twists counter clockwise 

So curl = $\nabla \times \vec{F}$ measures how much the vector field twists things

Claim: $\nabla \times \nabla f = \vec{0}$ **(in other words, curl of a vector field that comes from a gradient is zero)**

Another way to say it: conservative vector fields have zero twist!

Proof: just a calculation:

$$\nabla f = \vec{F} = (f_x, f_y, f_z) = (P, Q, R)$$

$$\nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \vec{k}$$

$$\nabla \times \vec{F} = \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) \vec{i} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right) \vec{j} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) \vec{k}$$

$$= (f_{zy} - f_{yz}) \vec{i} + (f_{xz} - f_{zx}) \vec{j} + (f_{yx} - f_{xy}) \vec{k}$$

$$= (0, 0, 0)$$ **By Clairaut's theorem**

So we have that if $\vec{F}$ comes from a scalar (there exists f such that $\nabla f = \vec{F}$) then $\nabla \times \vec{F} = 0$

And, though I won't prove it here: if $\nabla \times \vec{F} = 0$, then $\vec{F}$ is conservative

Something new: A new operator:

Suppose $\vec{F}$ **is a vector field in R³** $\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$

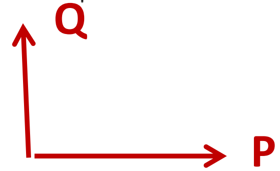
(at every point (x,y,z) in input space, it assigns a three dimensional vector)

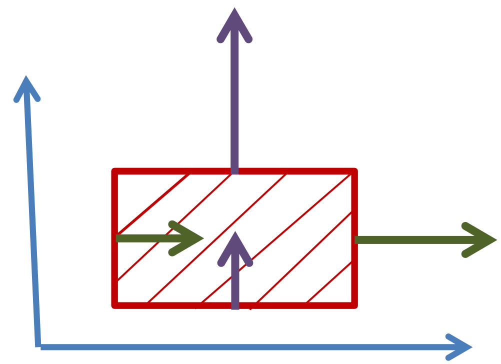
Then we can define the divergence: $\nabla \cdot \vec{F} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right)$ **Note: THIS IS A NUMBER!**

Example: $\vec{F} = (xz, xyz, -y^2)$ **Find** $\nabla \cdot \vec{F}$

Answer: $\nabla \cdot \vec{F} = \left(\frac{\partial (xz)}{\partial x}, \frac{\partial (xyz)}{\partial y}, \frac{\partial (-y^2)}{\partial z} \right) = z + xz + 0$

So what, geometrically, is divergence?





$\frac{\partial P}{\partial x} =$ **How much stuff flows in the x direction: if positive, then more comes out than goes in**

if negative, then more goes in than comes out

$\frac{\partial Q}{\partial y} =$ **How much stuff flows in the y direction: if positive, then more comes out than goes in**

if negative, then more goes in than comes out

Divergence is measuring whether the square balloon is expanding or contracting

Claim: $\nabla \cdot (\nabla \times \vec{F}) = 0$ **(in other words, the divergence of the curl of a vector field is zero**

Proof: Another calculation:

$$\nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0$$

All this leads to a fancy way to write Green's theorem: Suppose $\vec{F} = (P(x,y), Q(x,y), 0)$

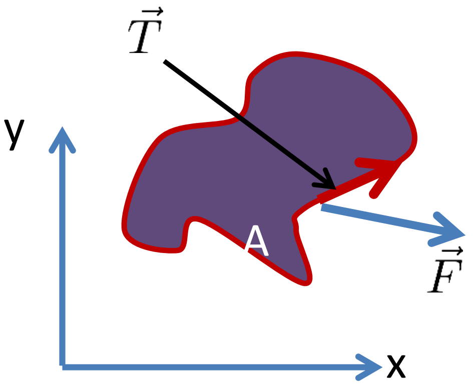
(no z component)

Then we have that $\nabla \times \vec{F} = (0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})$

So now we can write Green's theorem as

$$\int_C \vec{F} \cdot d\vec{r} = \int\int_A (\nabla \times \vec{F}) \cdot \vec{k} \; dA$$

Now, let's stand back a bit:



At any point on the boundary, we can decompose the vector field $\vec{F}$ into a normal and tangential component:

$$\vec{F} = [\vec{F}\cdot\vec{T}]\vec{\tau} + [\vec{F}\cdot\vec{n}]\vec{n}$$

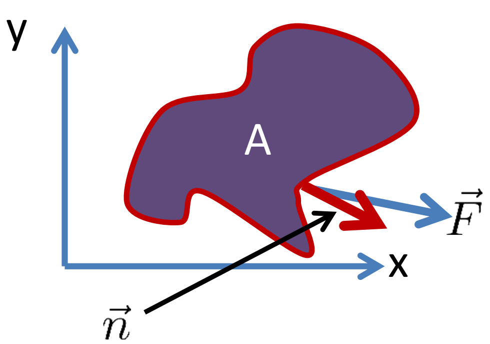
Vector field at a point on the boundary — tangential component — normal component

We can view Green's theorem as "collecting all the tangential components of $\vec{F}$ as we go around the boundary

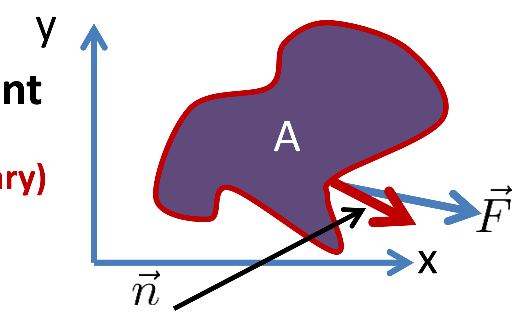
$$\int_C \vec{F}\cdot d\vec{r} = \int\int_A (\nabla\times\vec{F})\cdot\vec{k}\; dA \implies \int_C \vec{F}\cdot\vec{T}ds = \int\int_A (\nabla\times\vec{F})\cdot\vec{k}\; dA$$

What happens if we try to collect all the normal components? $\int_C \vec{F}\cdot\vec{n}ds$

Let's find out!



Collecting all the normal components $\int_C \vec{F} \cdot \vec{n}\, ds$



Step 1: Parameterize boundary to express the normal component

$\vec{r}(t) = x(t)\ \vec{i} + y(t)\ \vec{j}$ a ≤ t ≤ b **(goes all the way around boundary)**

So tangent is $\vec{r}\,'(t) = x\,'(t)\ \vec{i} + y\,'(t)\ \vec{j}$

So unit length tangent is $\vec{T}(t) = \frac{x\,'(t)}{|\vec{r}\,'(t)|}\ \vec{i} + \frac{y\,'(t)}{|\vec{r}\,'(t)|}\ \vec{j}$

Since $\vec{T} \cdot \vec{n} = 0$ **then** $\vec{n}(t) = \frac{y\,'(t)}{|\vec{r}\,'(t)|}\ \vec{i} - \frac{x\,'(t)}{|\vec{r}\,'(t)|}\ \vec{j}$

Step 2: Let's collect all the normals and see what we get:

$$\int_C \vec{F} \cdot \vec{n}\, ds = \int_a^b \left(\vec{F} \cdot \vec{n}(t)\right) |\vec{r}\,'|\, dt = \int_a^b \left[P \frac{y\,'(t)}{|\vec{r}\,'(t)|} - Q \frac{x\,'(t)}{|\vec{r}\,'(t)|}\right]\ |\vec{r}\,'(t)|\, dt$$

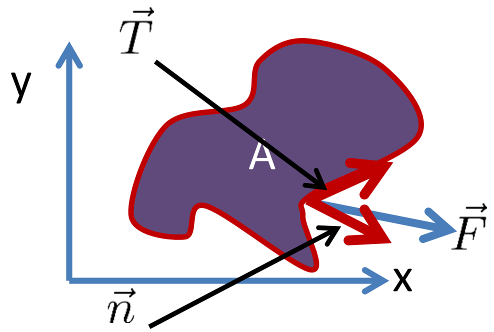
$$= \int_a^b [Py'(t)\ -\ Qx'(t)]\ dt$$

$$= \int_a^b Pdy\ -\ Qdx$$

Remember Green's theorem says: $\int_C \text{thing1}dx + \text{thing2}dy = \int \int_A \frac{\partial(\text{thing2})}{\partial x} - \frac{\partial(\text{thing1})}{\partial y}\, dxdy$

So, let thing1 = -Q and thing2 = P

$$= \int \int_A \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right] dxdy = \int \int_A \nabla \cdot \vec{F}\, dxdy$$ **WOW!!!**



If you collect all the tangent vectors: $\int_C \vec{F} \cdot \vec{T} ds = \int \int_A (\nabla \times \vec{F}) \cdot \vec{k} \; dA$

If you collect all the normal vectors: $\int_C \vec{F} \cdot \vec{n} ds = \int \int_A (\nabla \cdot \vec{F}) \; dA$

The Equivalence Loop:

(Caution: there are some caveats having to do with smoothness, connectivity of regions, etc. Check them!)

(1) The vector field $\vec{F}$ =P(x,y,z),Q(x,y,z),R(x,y,z) is the gradient of a scalar field f(x,y,z)

⇓

(2) The vector field F is conservative

⇓

(3) $\int_C \vec{F} \cdot dr = 0$ around any closed path C

⇓

(4) $\int_C \vec{F} \cdot dr$ is path independent

⇓

(5) $\nabla \times \vec{\mathbf{F}} = \mathbf{0}$

(5) ⇒ (1)

(1) If someone gives you an integral over a boundary, can you trade it for an integral over the interior?

(2) If someone gives you an integral over an interior, can you trade it for an integral over the boundary?

(3) If you are given an integral over a curve, is the curve closed? If so, can you apply the multidimensional fundamental theorem of calculus?

(4) Remember that chain of conservative—path integral zero—come from gradient---etc…can you use this?

(5) Given an integral in one coordinate system, is there a mapping that makes it easier somewhere else? (which might need the Jacobian)

(6) If you have to optimize something, and it has a constraint, how can you best set up a lagrange multiplier problem?