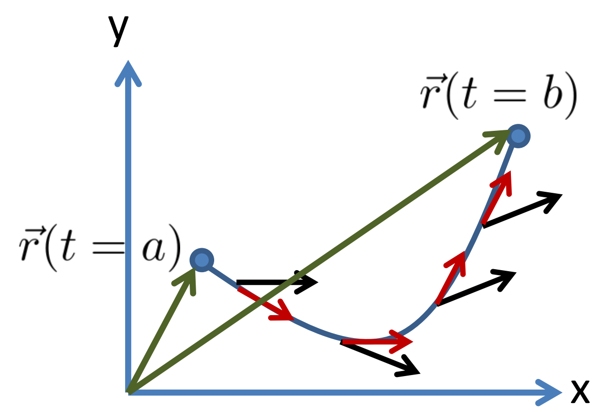
Section 16:4 Green's theorem



For 1D calculus, we had that $\int_a^b \frac{df}{dx}dx = f(b) - f(a)$

[The integral of the derivative of a function is just the difference between its values at the two endpoints]

Theorem: In more dimensions, we have that

$$\int_C \nabla f \cdot d\vec{r} = f[\vec{r}(t=b)] - f[\vec{r}(t=a)]$$

Conservative: There exists a scalar f(x,y) such that $\nabla f = \vec{F}(x,y)$

⇕

The vector field $\vec{F} = (P,Q)$ **is path-independent**

⇕

So now we've got:
Given a vector field (P,Q)

$\int_C \vec{F} \cdot d\vec{r} = 0$ **Over any closed curve C**

⇕

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Conservative: There exists a scalar f(x,y) such that $\nabla f = \vec{F}(x,y)$

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Given a vector field (P,Q)

$\int_C \vec{F} \cdot d\vec{r} = 0$ **Over any closed curve C**

⇕

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Example: show that the vector field $\vec{F} = (xy^2, yx^2)$ **is path-independent**

$P_y = 2xy$ $Q_x = 2xy$ so yes!!!

Example: Suppose (P,Q) = (3+2xy, x^2-3y^2).

(1) Is there a function f(x,y) such that $\vec{F} = (P, Q) = \nabla f$

(2) Find that function f(x,y)

(3) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ **over the curve** $C = \vec{r}(t) = e^t \sin t\ \vec{i} + e^t \cos t\ \vec{j},\ 0 \leq t \leq \pi$

Answer to #1

We check to see if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ $\quad \frac{\partial P}{\partial y} = 2x$ and $\frac{\partial Q}{\partial x} = 2x$ ✓

Answer to #2

We try to build f(x,y) through integration. We require that f_x=P and f_y = Q

f_x = P ⟹ $\int f_x dx = \int P(x,y)dx$ ⟹ $f(x,y) = \int (3 + 2xy)dx = 3x + x^2y + g(y)$

So $f_y = x^2 + \frac{dg(y)}{dy}$ **which must satisfy f_y = Q which =x^2-3y^2 so dg(y)/dy=-3y^2 so g(y) = -y^3 so f(x,y) = 3x + x^2y – y^3 +C**

Since the vector field comes from a gradient, then we need only evaluate at endpoints

$$\int_C \nabla f \cdot d\vec{r} = f[\vec{r}(t = b)] - f[\vec{r}(t = a)] = f(0, -e^\pi) - f(0, 1)$$

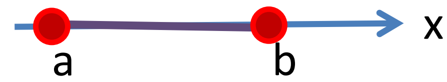
$$= (3x + x^2y - y^3)\Big|_{(0,-e^\pi)} - (3x + x^2y - y^3)\Big|_{(0,1)} = e^{3\pi} - (-1)$$

So, let's think about these things in terms of the "dimensions in which they live"

1D input space:
f(x): $R^1 \rightarrow R^1$
Blue axis

Integrate over 1D object (purple line segment)

Boundary is a 0D object

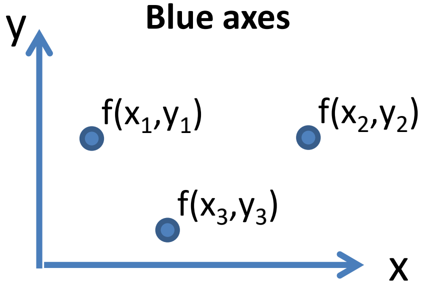


Fundamental theorem

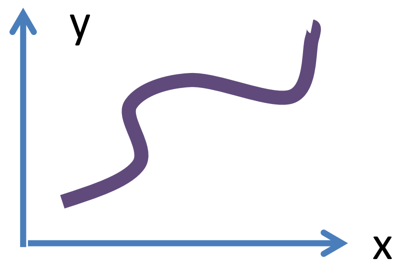
$$\int_a^b \frac{df}{dx} dx = f(b) - f(a)$$

Integral of derivative over object = function over object boundary

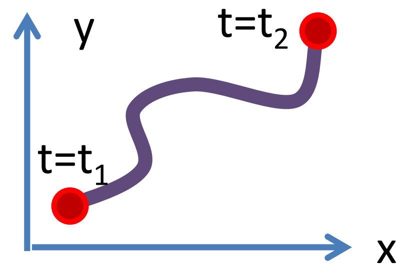
2D input space:
f(x,y): $R^2 \rightarrow R^1$
Blue axes



Integrate over 1D domain (purple curve)



Boundary is a 0D object



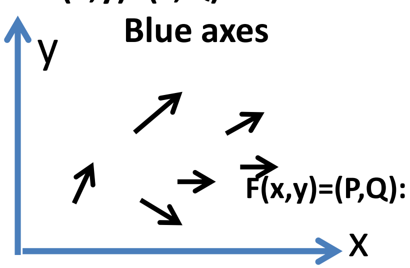
Fundamental theorem

$$\int_{t_1}^{t_2} \nabla f \cdot d\vec{r}$$
$$= f[\vec{r}(t_2)] - f[\vec{r}(t_1)]$$

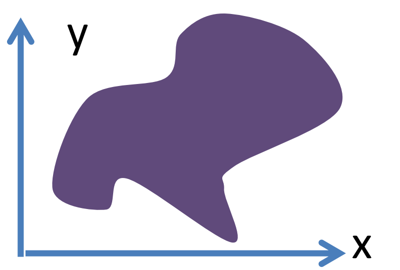
Integral of derivative over object equals function over boundary

Can we go higher?

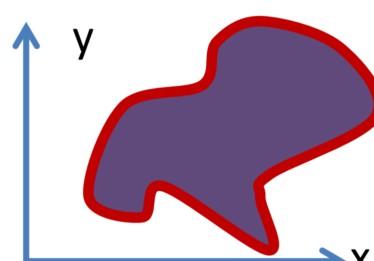
2D input space:
$\vec{F}(x,y)=(P,Q)$: $R^2 \rightarrow R^2$
Blue axes



Integrate over 2D domain (purple region)

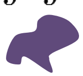


Boundary is a 1D domain



A fundamental theorem?

$\int\int$ **[derivative of $\vec{F}$] dxdy**



=

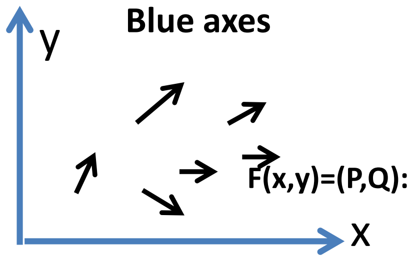
$\int$ **vector field $\vec{F}$]** $\cdot\, dC$

Integral of "derivative" over object equals function over boundary

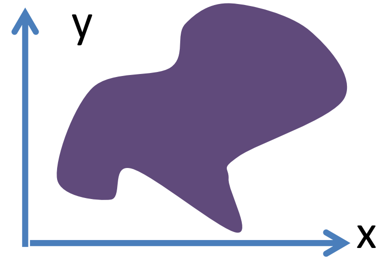
YES!!! Green's Theorem

Can we go higher?

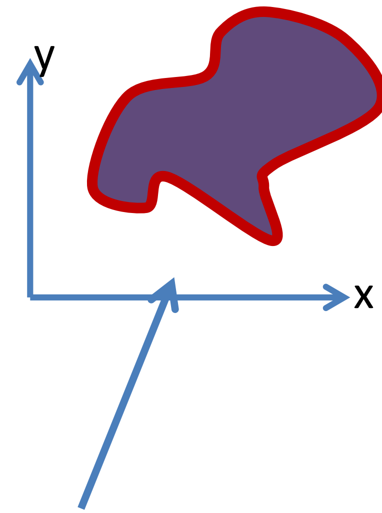
2D input space:
$\vec{F}(x,y)=(P,Q)$: $R^2 \rightarrow R^2$
Blue axes



Integrate over 2D domain
(purple region)



Boundary is a 1D domain



A fundamental theorem?

$\int\int$ [derivative of $\vec{F}$] dxdy

=

$\int$ vector field $\vec{F}$] $\cdot\, dC$

Integral of "derivative" over object equals function over boundary

YES!!! Green's Theorem

Time for Green's theorem

Claim: Suppose $\vec{F} = (P,Q) = P\vec{i} + Q\vec{j}$

Let C be closed curve surrounding a region A

Then:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy = \int\int_A \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dxdy$$

Wow!

The line integral of $\vec{F} \cdot d\vec{r}$ **over the boundary equals the double integral of**

$$\left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]$$ **over the interior**

Green's Theorem

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy = \int \int_A \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dx dy$$

Recall the loop of equalities:

Conservative: There exists a scalar f(x,y,z) such that $\nabla f = \vec{F}(x,y,z)$

⇕

The vector $\vec{F} = (P, Q)$ **path-independent**

⇕

$\int_C \vec{F} \cdot d\vec{r} = 0$ **Over any closed curve C**

⇕

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

I hope you see why this is so satisfying:

(1) If $\int_C \vec{F} \cdot d\vec{r} = 0$ **then the above loop says** $\left[\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \right]$ **which works with Green's**

(2) If $\left[\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \right]$ **then the area integral integrates to zero, so that means** $\int_C \vec{F} \cdot d\vec{r} = 0$

Green's Theorem

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy = \int \int_A \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dx dy$$

Before proving it, let's use it:

Example 1: **Let C be the unit circle. Let** $\vec{F} = \frac{-y}{2} \vec{i} + \frac{x}{2} \vec{j}$ **Find** $\int_C \vec{F} \cdot d\vec{r}$

(the hard way:) let's do the path integral $\int_C \vec{F} \cdot d\vec{r}$

(1) Parameterize the unit circle: C(t) = (cos t, sin t), 0 ≤ t ≤ 2 π

(2) We have that $\vec{r}(t) = (\cos t, \sin t)$, so $\vec{r}'(t) = (-\sin t, \cos t)$

(3) So $\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(\frac{-y}{2}, \frac{x}{2} \right) \cdot \vec{r}'(t) \, dt$

$$= \int_0^{2\pi} \left(\frac{-\sin t}{2}, \frac{\cos t}{2} \right) \cdot (-\sin t, \cos t) = \frac{1}{2} \int_0^{2\pi} \left(\sin^2 t + \cos^2 t \right) dt$$

$$= \frac{1}{2} \int_0^{2\pi} dt = \pi$$

(the easy way:) let's do the area integral $\int \int_A \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dx dy$

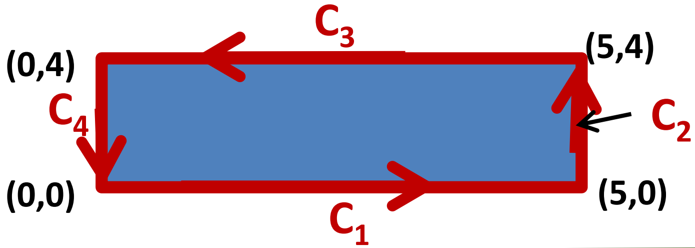
$$\int \int_A \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dx dy = \int \int_A \left[\frac{1}{2} - \frac{-1}{2} \right] dx dy = \int \int_{unitdisk} 1 dx dy = \pi (1)^2$$

Green's Theorem

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy = \int \int_A \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dx dy$$

Example 2: Another one: Let C bound the rectangle:

$\vec{F} = y^2 \vec{i} + x^2 y \vec{j}$ **Find** $\int_C \vec{F} \cdot d\vec{r}$



(the Green's theorem way:) let's do the area integral $\left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] = 2xy - 2y$

$$\int \int_A \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dx dy = 2 \int_0^5 \int_0^4 (xy - y) dy dx = 2 \int_0^5 \left[\Big|_0^4 \left[\frac{xy^2}{2} - \frac{y^2}{2} \right] \right] dx = 2 \int_0^5 (8x - 8) dx$$

$$= 16 \int_0^5 (x - 1) dx = 16 \Big|_0^5 (x^2/2 - x) dx = 16 (\tfrac{25}{2} - 5) = (200 - 80) = 120$$

(the path integral way)

$$\int_C \vec{F} \cdot d\vec{r} = \int_C [P dx + Q dy] = \int_{C_1} P dx + \int_{C_2} Q dy + \int_{C_3} P dx + \int_{C_4} Q dy$$

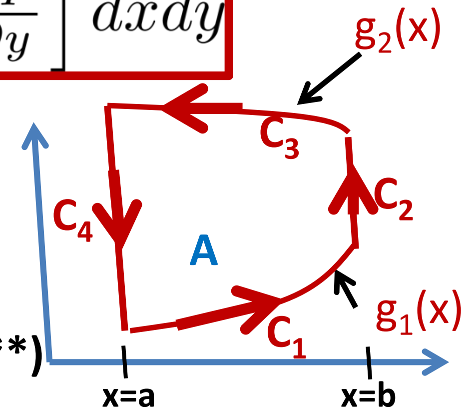
$$= \int_{C_1} y^2 dx + \int_{C_2} x^2 y dy + \int_{C_3} y^2 dx + \int_{C_4} x^2 y dy$$

= [0 because y=0] $+ \int_0^4 25 y dy + \int_5^0 16 dx +$ **[0 because x=0]**

$$= \Big|_0^4 25y^2/2 + \Big|_5^0 16x = 25 * 16/2 + (-80) = 120$$

Green's Theorem

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy = \int \int_A \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dx dy$$



Hard to prove in general: let's do it for region with $g_1(x)$ bottom curve and $g_2(x)$ top curve

(1) If we can show that

$$\int_C P dx = \int \int_A -\frac{\partial P}{\partial y} dx dy \quad \textbf{(Eqn.*) and} \quad \int_C Q dy = \int \int_A \frac{\partial Q}{\partial x} dx dy \quad \textbf{(Eqn.**)}$$

then we are done (hint: add them together and you get Green's theorem)

(2) **First we do the area integral in Eqn. *:**

$$\int \int_A -\frac{\partial P}{\partial y} dx dy = \int_{x=a}^{x=b} \left[\int_{g_1(x)}^{g_2(x)} -\frac{\partial P}{\partial y} dy \right] dx = -\int_a^b \left[P(x, g_2(x)) - P(x, g_1(x)) \right] dx$$

(3) **Now we do the path integral in Eqn. *:**

$$\int_C P dx = \int_{C_1} P dx + \int_{C_2} P dx + \int_{C_3} P dx + \int_{C_4} P dx$$

$$\int_a^b P(x, g_1(x)) dx + \int_b^a P(x, g_2(x)) dx = \int_a^b \left[P(x, g_1(x)) - P(x, g_2(x)) \right] dx$$

Second and fourth integrals are zero, since dx = 0 on those sides

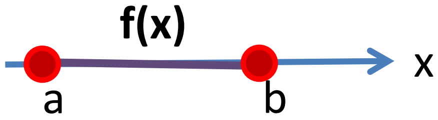
Equal!

(4) Similar proof for Eqn. ** shows the other one. So proved!

16.5: Curl and Divergence

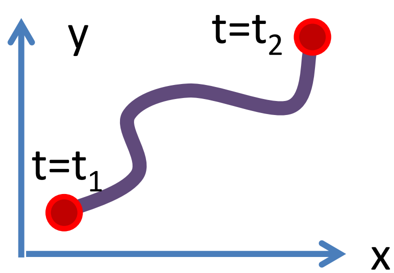
So far, we have established three theorems that relate the integral of the derivative of a function over a region to the values of the function on the boundary:

Boundary is a 0D object



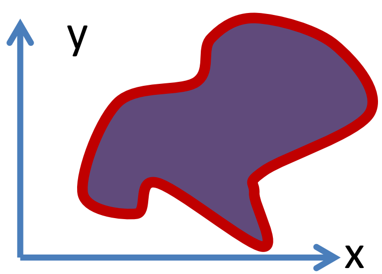
$$\int_a^b \frac{df}{dx}dx = f(b) - f(a)$$

Boundary is a 0D object



$$\int_{t_1}^{t_2} \nabla f \cdot d\vec{r} = f[\vec{r}(t_2)] - f[\vec{r}(t_1)]$$

Boundary is a 1D domain



$$\int_C \vec{F} \cdot d\vec{r} = \int\int_A \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dxdy$$

Amazingly, we can do more –but we need a few more definitions of "operators"

Curl and Divergence

Suppose $\vec{F}$ is a vector field in R³ $\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$

(at every point (x,y,z) in input space, it assigns a three dimensional vector)

We define the curl of $\vec{F}$ to be the vector $\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$

How is anyone supposed to remember this? **Note: THIS IS A VECTOR!**

Here's how:

(1) Recall that $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{\partial f}{\partial x}\ \vec{i} + \frac{\partial f}{\partial y}\ \vec{j} + \frac{\partial f}{\partial z}\ \vec{k}\right)$

(2) We can then think of ∇ as a vector operator with components

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\vec{i}\ \frac{\partial}{\partial x} + \vec{j}\ \frac{\partial}{\partial y} + \vec{k}\ \frac{\partial}{\partial z}\right)$$

(3) So now we can define the curl as the cross product between ∇ and $\vec{F}$

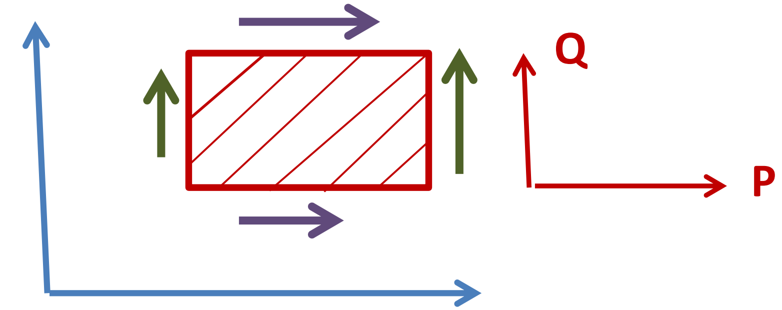
$$\nabla \times \vec{F} \equiv \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y,z) & Q(x,y,z) & R(x,y,z) \end{bmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\ \vec{i}\ + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\ \vec{j}\ + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ \vec{k}$$

So what is curl geometrically? Here goes…..

Let's look at a simpler vector field: $\vec{F} = (P, Q, R) = (P(x,y), Q(x,y), 0)$ **(the z component is zero)**

Then: $\nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \vec{k} = \left(0, 0, \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right)$



Let's draw a box in the xy plane

$\frac{\partial P}{\partial y} =$ **the twisting of the box due to P**

If P is bigger on top than the bottom, then it twists clockwise

If P is smaller on top than the bottom, then it twists counter clockwise

$\frac{\partial Q}{\partial x} =$ **the twisting of the box due to Q**

If Q is bigger on left than the right, then it twists clockwise

If Q is smaller on left than the right, then it twists counter clockwise

So curl = $\nabla \times \vec{F}$ measures how much the vector field twists things

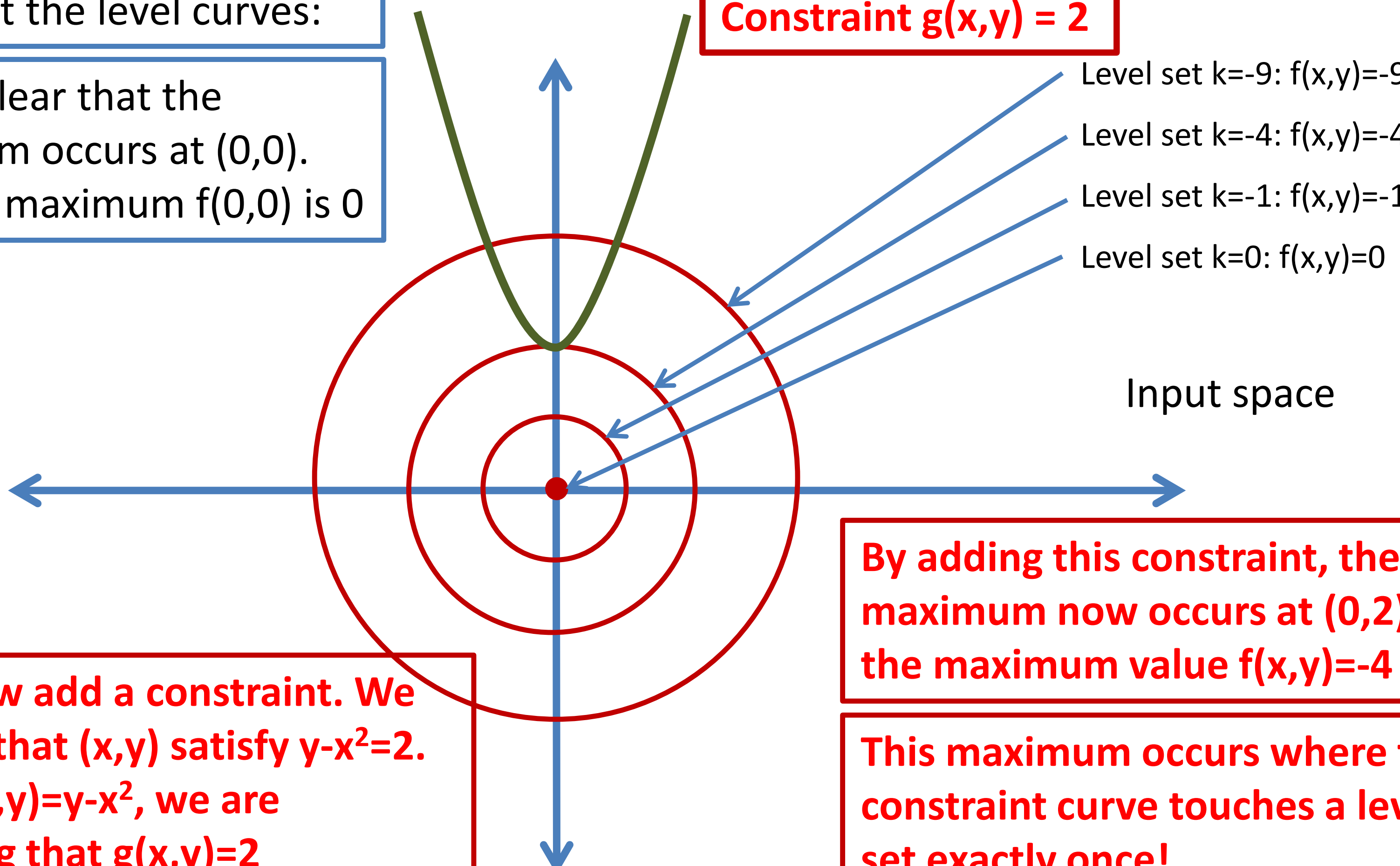
Lagrange Multipliers

Want to maximize f(x,y) = - [x^2 +y^2]

Let's plot the level curves:

Seems clear that the maximum occurs at (0,0). And the maximum f(0,0) is 0

Constraint g(x,y) = 2



By adding this constraint, the maximum now occurs at (0,2), and the maximum value f(x,y)=-4

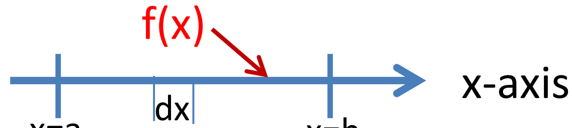
Let's now add a constraint. We require that (x,y) satisfy $y-x^2=2$. So if $g(x,y)=y-x^2$, we are requiring that g(x,y)=2

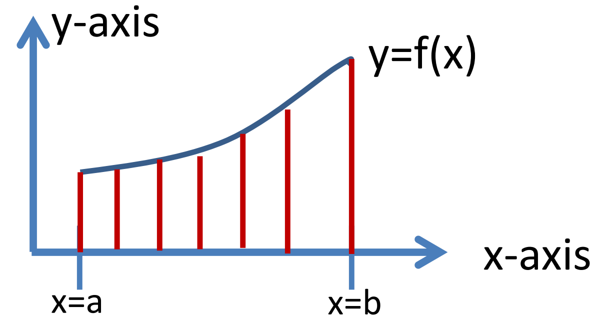
This maximum occurs where the constraint curve touches a level set exactly once!

Extreme value occurs where normal to the constraint curve points in the direction as the normal to the level set

Chapter 15: Integration

1D Integration: f(x)

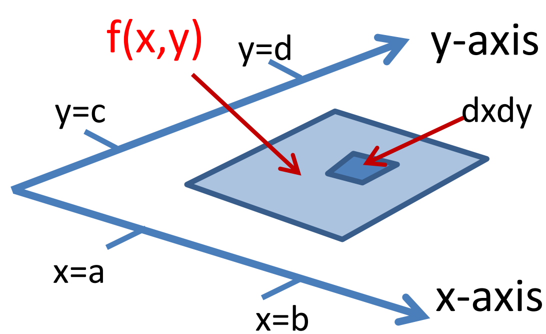


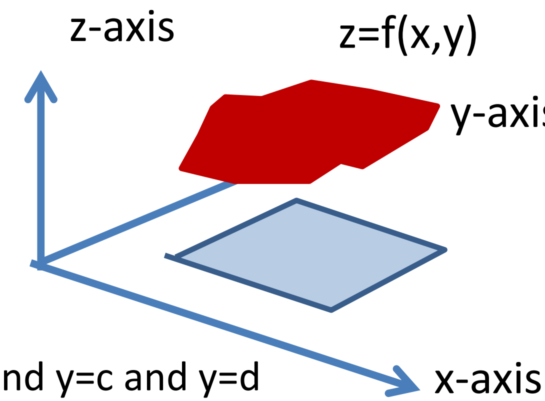


$\int_{x=a}^{x=b} f(x)dx$ = amount of f(x) between x=a and x=b

= area under curve y=f(x) between x=a and x=b

2D Integration: f(x,y)

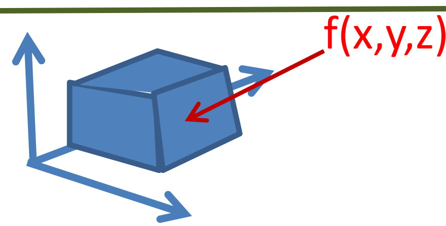


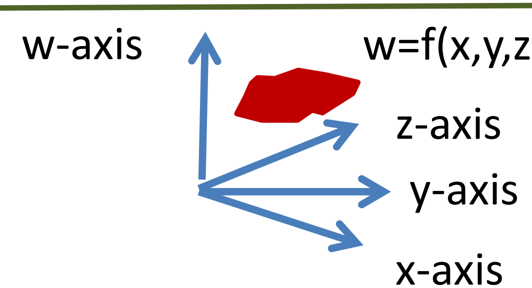


$\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y)\ dxdy$ = amount of f(x,y) between x=a and x=b and y=c and y=d

= volume under surface z=f(x,y) between x=a and x=b and y=c and y=d

3D Integration: f(x,y,z)

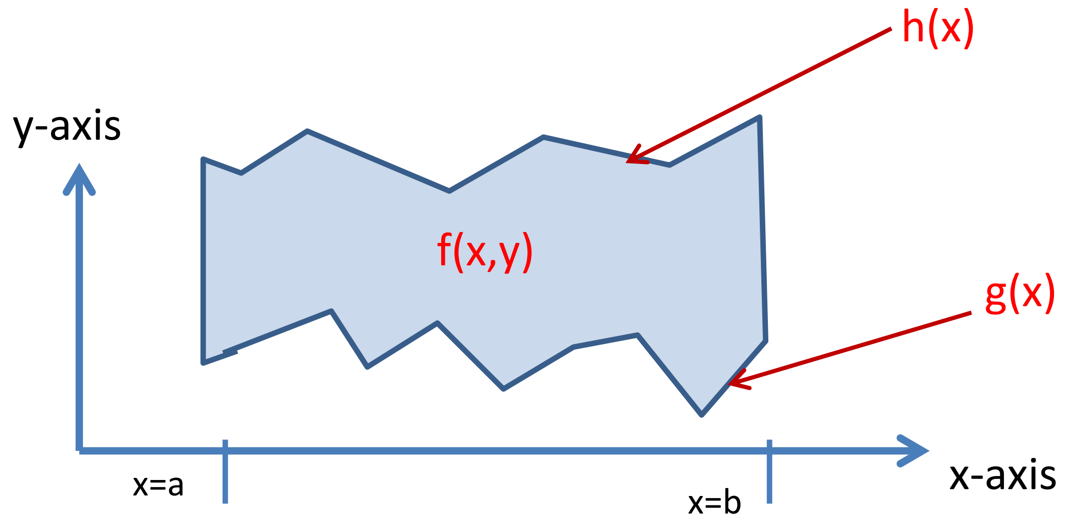




$\int_{z=e}^{z=f} \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y)\ dxdydz$ = amount of f(x,y,z) between x=a and x=b and y=c and y=d and z=e and z=f

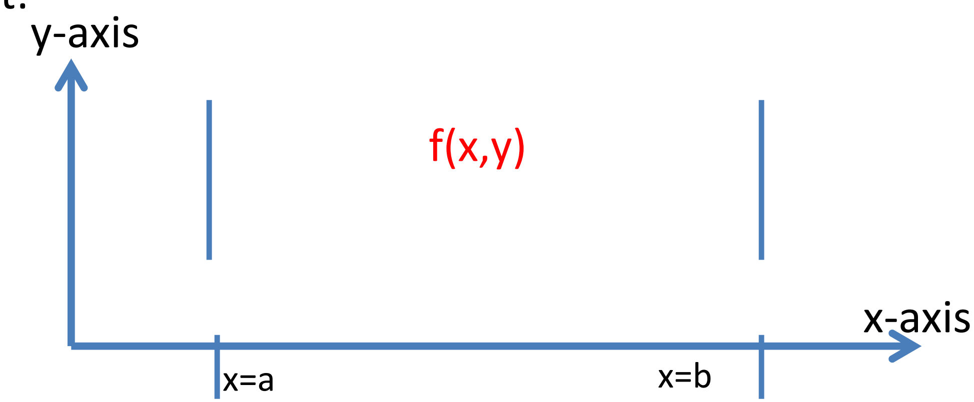
= volume under graph z=f(x,y,z) between x=a and x=b and y=c and y=d and z=e and z=f

Integration over weird regions



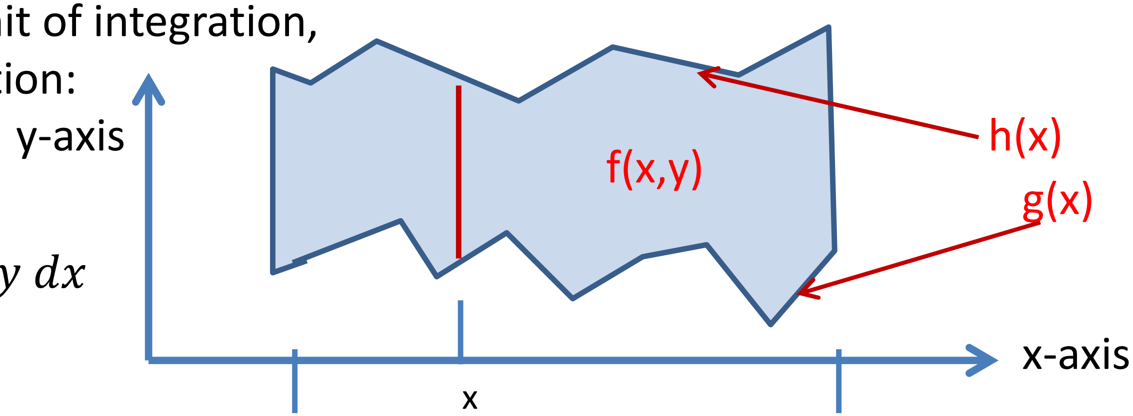
Choose outer limits of integration first:

$$\int_{x=a}^{x=b}\int_{?}^{?} \quad dx$$



For each value of the outer limit of integration, find the inner limits of integration:

$$\int_{x=a}^{x=b}\int_{g(x)}^{h(x)} f(x,y)dy\, dx$$

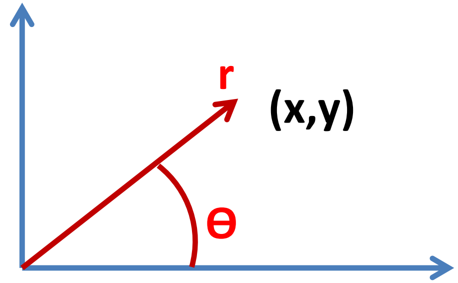


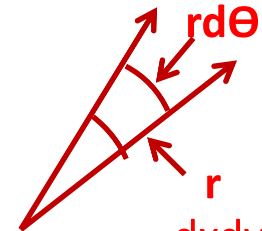
Other coordinate systems:

Polar coordinates:

$x = r\cos(\theta)$

$y = r\sin(\theta)$





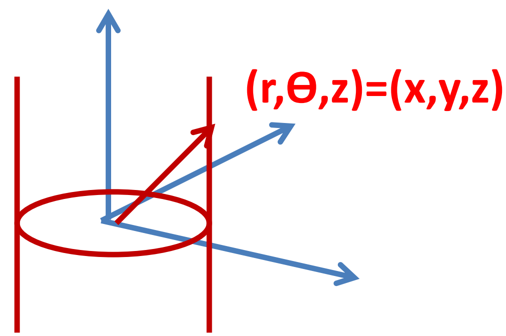
$dxdy = r\,dr\,d\theta$

Cylindrical coordinates:

$x = r\cos(\theta)$

$y = r\sin(\theta)$

$z = z$



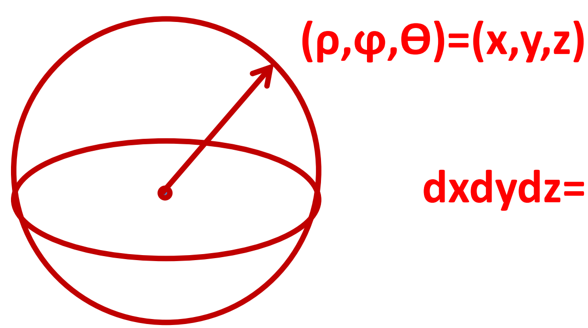
$dxdydz = r\,dr\,d\theta\,dz$

Spherical coordinates:

$x = \rho\sin(\varphi)\cos(\theta)$

$y = \rho\sin(\varphi)\sin(\theta)$

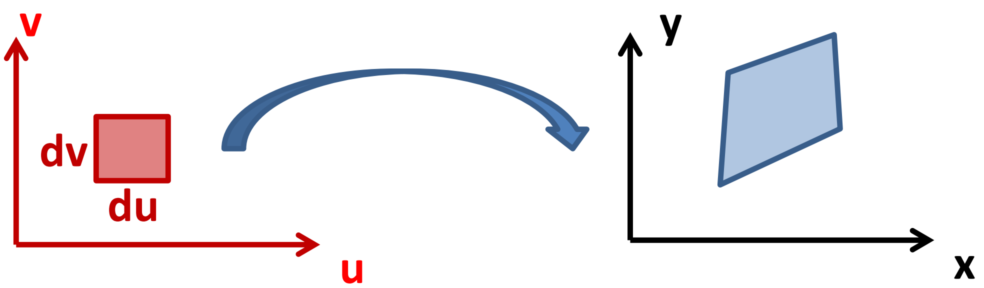
$z = \rho\cos(\varphi)$



$dxdydz = \rho^2\sin(\emptyset)\,d\rho\,d\varphi\,d\theta$

Change of Variable:

x=x(u,v)
y=y(u,v)



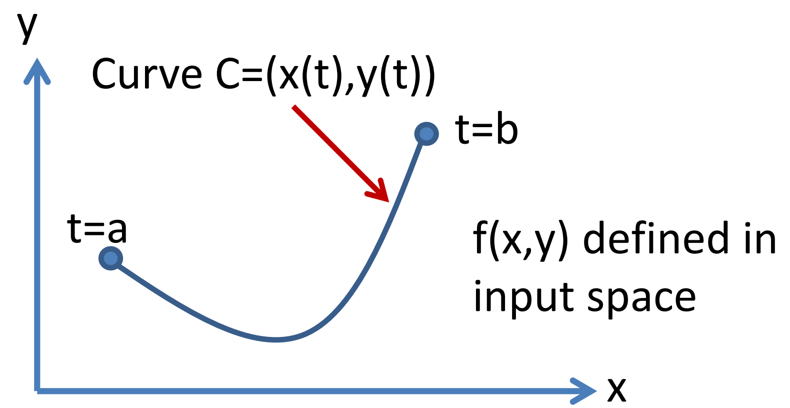
What is dxdy? (What is the magic factor?)

$$dxdy = \left| \begin{matrix} \frac{\partial x}{du} & \frac{\partial x}{dv} \\ \frac{\partial y}{du} & \frac{\partial y}{dv} \end{matrix} \right| dudv = J(u,v)\, dudv$$

$$\iint_{x1.y1}^{x2.y2} f(x,y)dxdy = \iint_{u1,v1}^{u2,v2} f\big(x(u,v),y(u,v)\big)J(u,v)du\,dv$$

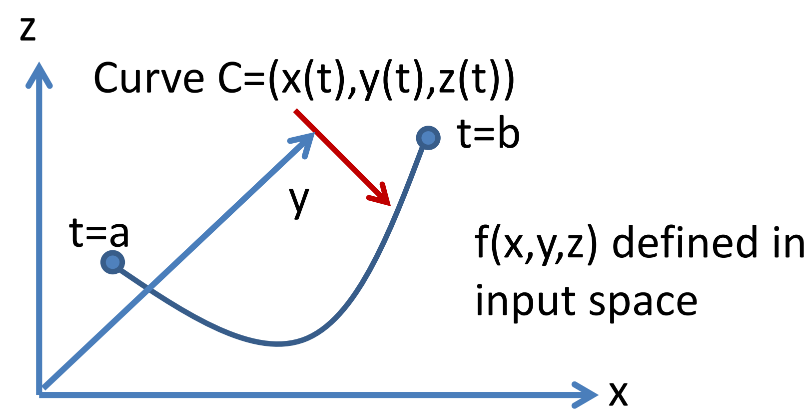
Chapter 16: Integration and the fundamental theorem of calculus

Line integrals of a scalar function $f:R^2$ to R



$$\int_C f(x,y)ds = \int f(x,y)\left[\left(\frac{\partial x}{dt}\right)^2 + \left(\frac{\partial y}{dt}\right)^2\right]^{(1/2)} dt$$

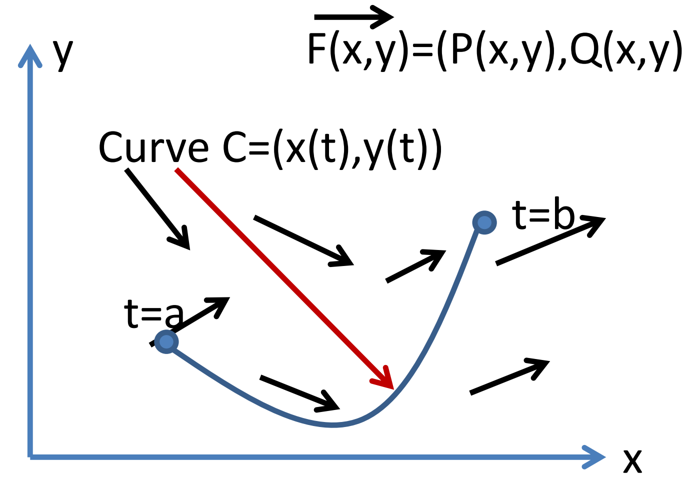
Line integrals of a scalar function $f:R^3$ to R



$$\int_C f(x,y,z)ds = \int f(x,y,z)\left[\left(\frac{\partial x}{dt}\right)^2 + \left(\frac{\partial y}{dt}\right)^2 + \left(\frac{\partial z}{dt}\right)^2\right]^{(1/2)} dt$$

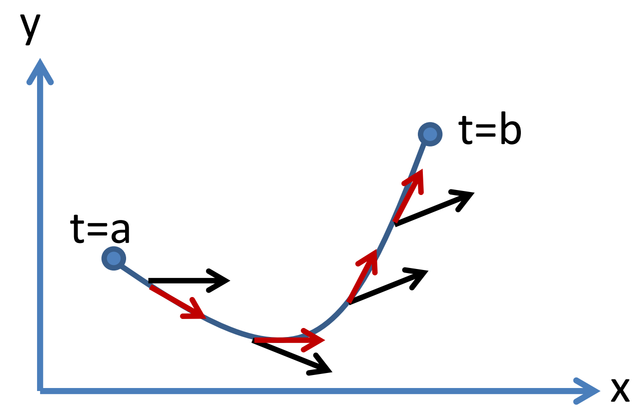
Chapter 16: Integration and the fundamental theorem of calculus

Line integrals over a vector field from R^2 to R^2



Collect all the **tangential components** $\boldsymbol{\tau}$ of $\vec{F}$ along C

$$\int_C F \cdot d\tau = \int_c \boldsymbol{Pdx} + \boldsymbol{Qdy}$$



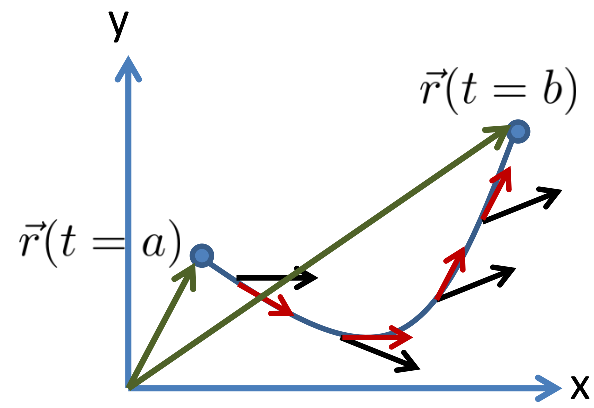
3D: Collect all the **tangential components** $\boldsymbol{\tau}$ of F along C

$$\int_C F \cdot d\tau = \int_c \boldsymbol{Pdx} + \boldsymbol{Qdy} + \boldsymbol{Rdz}$$

And finally

Theorem: In more dimensions, we have that

$$\int_C \nabla f \cdot d\vec{r} = f[\vec{r}(t=b)] - f[\vec{r}(t=a)]$$



Conservative: There exists a scalar f(x,y,z) such that $\nabla f = \vec{F}(x,y,z)$

⇕

The vector field $\vec{F} = (P, Q)$ **is path-independent**

⇕

So now we've got:
Given a vector field (P,Q)

$\int_C \vec{F} \cdot d\vec{r} = 0$ **Over any closed curve C**

⇕

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$