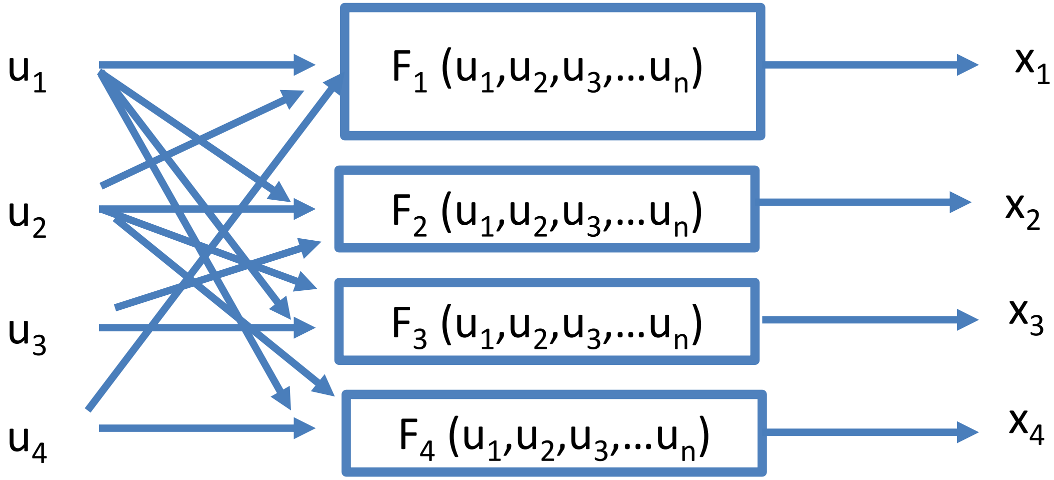
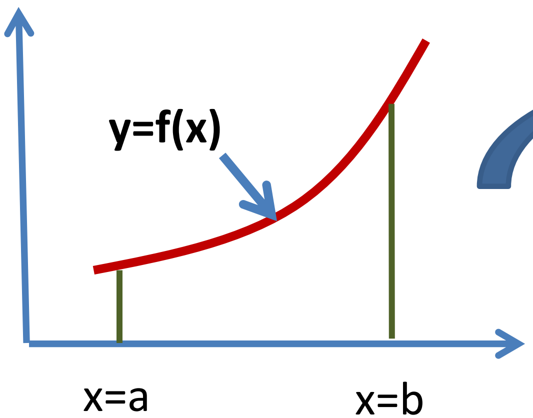


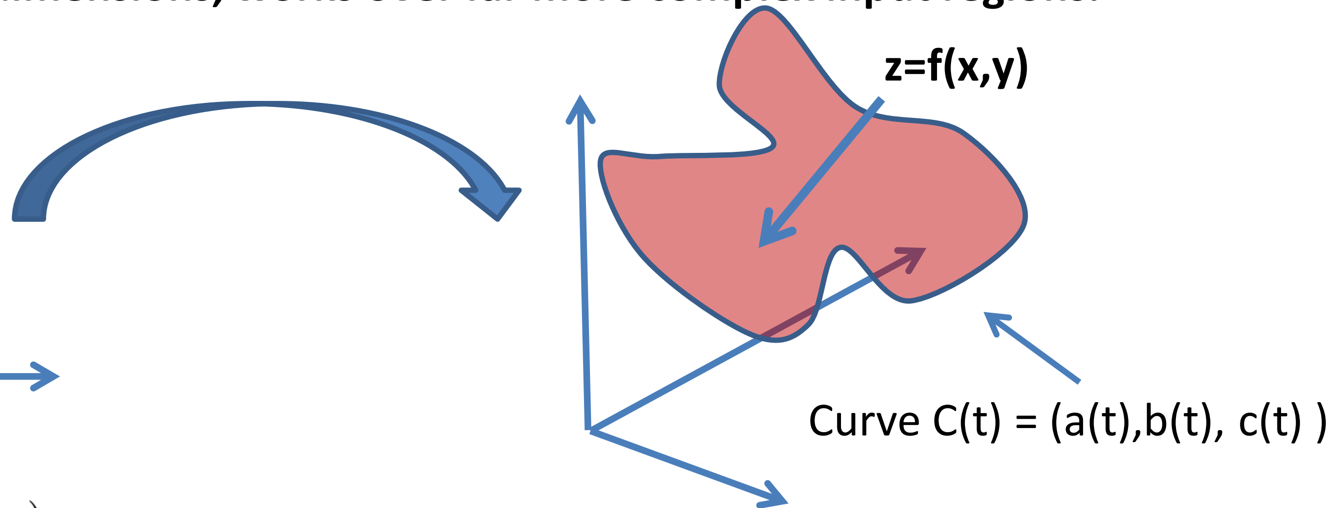
Jacobian =

$$\begin{matrix} \nabla f_1 \\ \nabla f_2 \\ \nabla f_3 \\ \nabla f_4 \end{matrix}$$

Section 16: Culmination of the Course

This is the chapter when you learn that the fundamental theory of calculus applies in multi-dimensions, works over far more complex input regions.





$$\int_a^b \frac{df}{dx}dx = f(b) - f(a)$$

1D Fundamental Theorem Calculus

"integral of a derivative of f(x) over a region"

=

"difference of function over boundary"

Multi-Dimensional Fundamental Theorem Calculus

integral of a "derivative of f(x,y) over a region"

=

difference of function over boundary

What do any of these words: "derivative of f(x,y)" and "difference of function" mean?

Section 16:1 We will assemble these ideas in steps…

We begin by defining a vector field:

Definition Let *D* be a set in R^2 (a plane region). A **vector field** $\vec{F}(x,y)$ **on** R^2 is a function F that assigns to each point (*x*, *y*) in *D* a two-dimensional vector

Examples:

F(x,y)= (3x,3y)

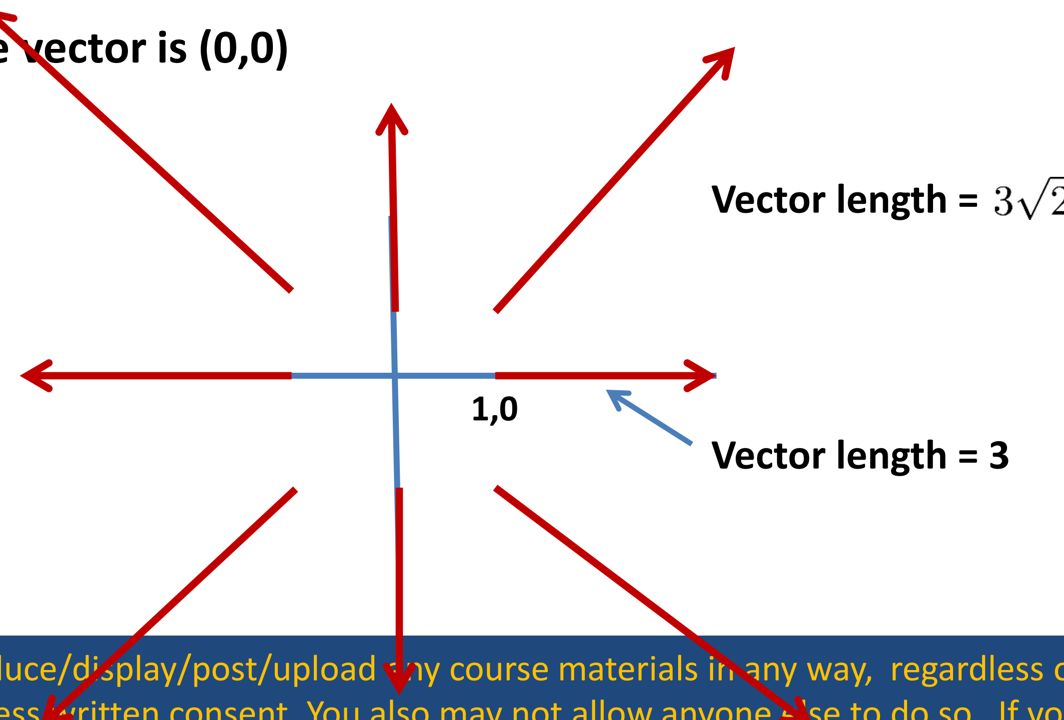
So at input (1,0), the vector is (3,0) **So at input (-1,0), the vector is (-3,0)**

So at input (0,1), the vector is (0,3) **So at input (0.-1), the vector is (0,-3)**

So at input (1,1), the vector is (3,3) **So at input (-1.-1), the vector is (-3,-3)**

So at input (0,0), the vector is (0,0)

Let's draw some vectors in this vector field

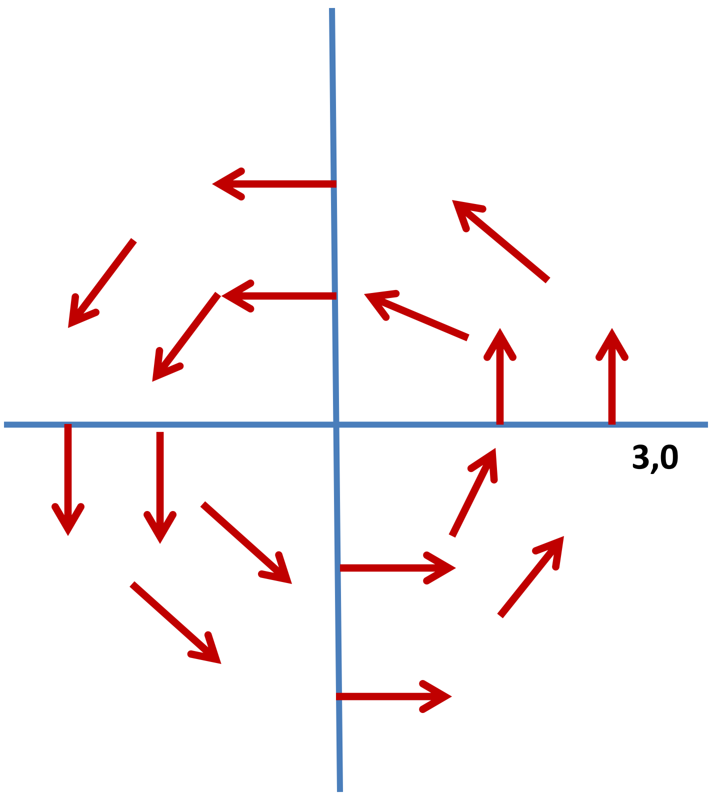


Section 16:1 We will assemble these ideas in steps…

Example: $\vec{F}(x,y) = \frac{-y,x}{\sqrt{(x^2+y^2)}}$

All vectors have length 1

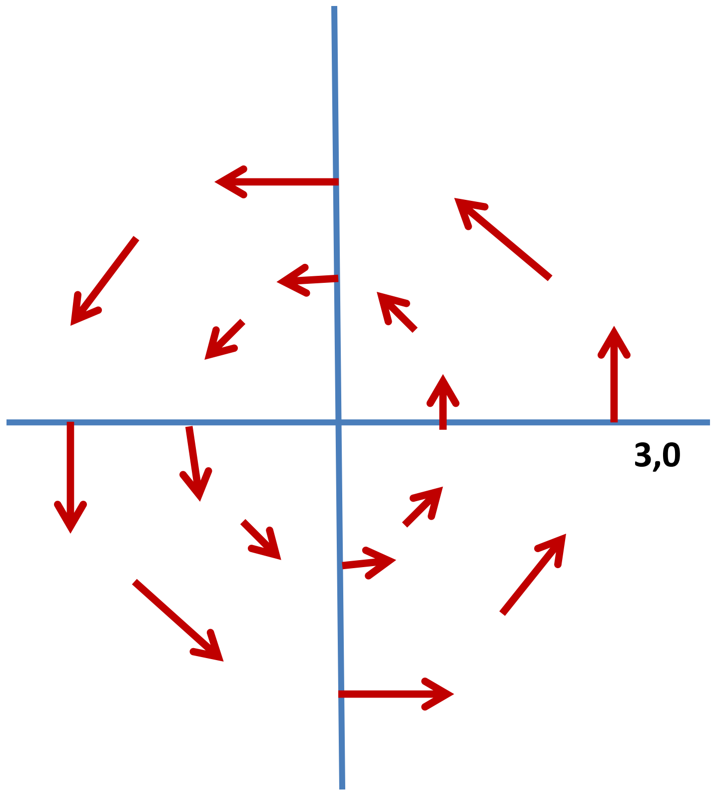
Let's draw some vectors in this vector field



$\vec{F}(x,y) = (-y,x)$

Vectors get longer the further they are from (0,0)

Let's draw some vectors in this vector field



Section 16:1 We will assemble these ideas in steps…

A real life example:

From: https://telescoper.wordpress.com/2018/10/12/vector-calculus-weather/

Section 16:1 We will assemble these ideas in steps…

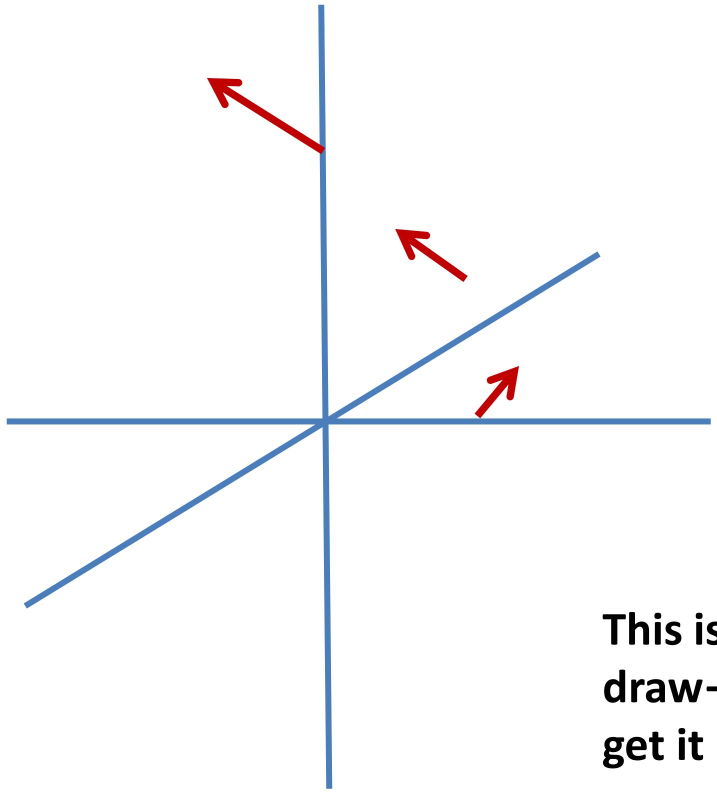
The same idea holds in 3D:

Definition Let *D* be a set in R^3. A **vector field on** R^3 is a function $\vec{F}$ that assigns to each point (*x y,z*) a three-dimensional vector $\vec{F} = (F_1, F_2, F_3)$

Examples:

$$\vec{F}(x, y, z) = (-y, x, z)$$

We just found that the first two components spin around the origin in the x-y plane



So it's the same, except the z component gets steeper and steeper

This is hard to draw—I hope you get it

Section 16:1 We will assemble these ideas in steps…

Two more definitions---which will be really useful later….

Definition: Given a function f(x,y), we define the gradient field as $\nabla f = (f_x, f_y)$

(same thing in 3D or more)

Definition: Given a function f(x,y,z), we define the gradient field as $\nabla f = (f_x, f_y, f_z)$

Example: if f(x,y,z)= x^2 y^3 sin z, find $\nabla f = (f_x, f_y, f_z)$

$$\nabla f = (2xy^3 \sin z, x^2(3y^2) \sin z, x^2 y^3 \cos z)$$

Definition: we say that a vector field $\vec{F}$ is conservative if there exists a scalar function f such that $\nabla f = \vec{F}$

Example: the vector field $\vec{F} = (2xy^3 \sin z, x^2(3y^2) \sin z, x^2 y^3 \cos z)$ is conservative

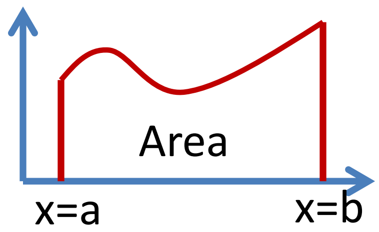
Because there exists a function f such that $\nabla f = \vec{F}$

Section 16:2 We will assemble these ideas in steps…

Begin by remembering what we meant by a one-dimensional integal:

Question: Find the area under the curve y=f(x) between x=a and x=b

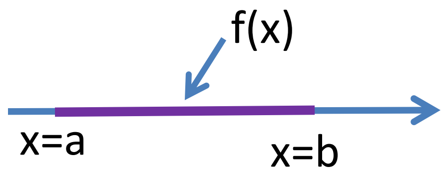
This one wants you to integrate the area under a function graphed with output against input



$$\text{Area} = \int_a^b f(x)dx$$

Question: Integrate the function f(x) along the x axis between x=a and x=b

This one wants you to integrate the total "weight" of a function describe in input space.



$$\text{Answer} = \int_a^b f(x)dx$$

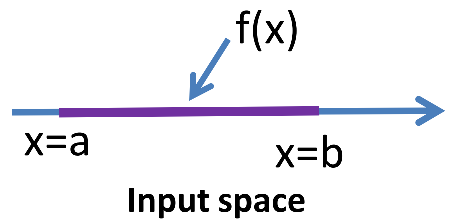
Note that they are the same thing…

We will start with this second interpretation:

Section 16:2 We will assemble these ideas in steps…

Question: Integrate the function f(x) along the x axis between x=a and x=b

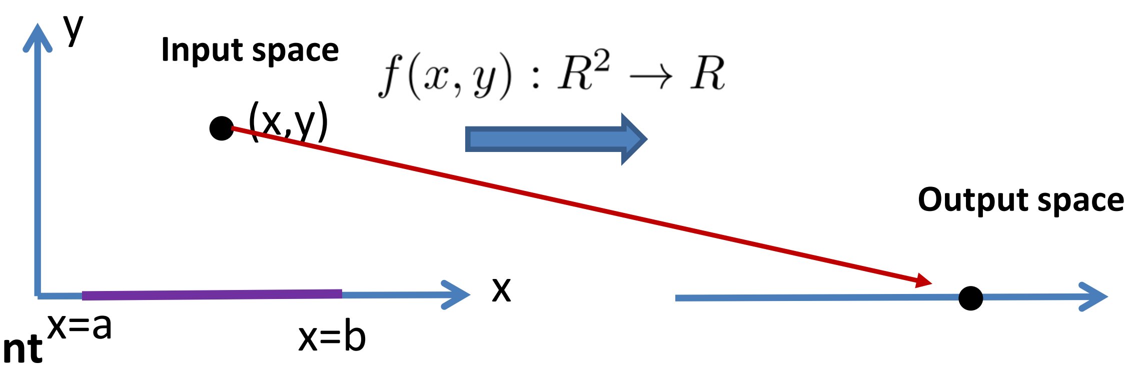
This one wants you to integrate the total "weight" of a function describe in input space.



$\text{Answer} = \int_a^b f(x)dx$

Suppose I give you a function f(x,y) $f(x,y) : R^2 \rightarrow R$

Now, suppose I ask you to integrate f(x,y) along the line segment from (a,0) to (b,0)



In other words, find the total amount of f(x,y) along that purple segment

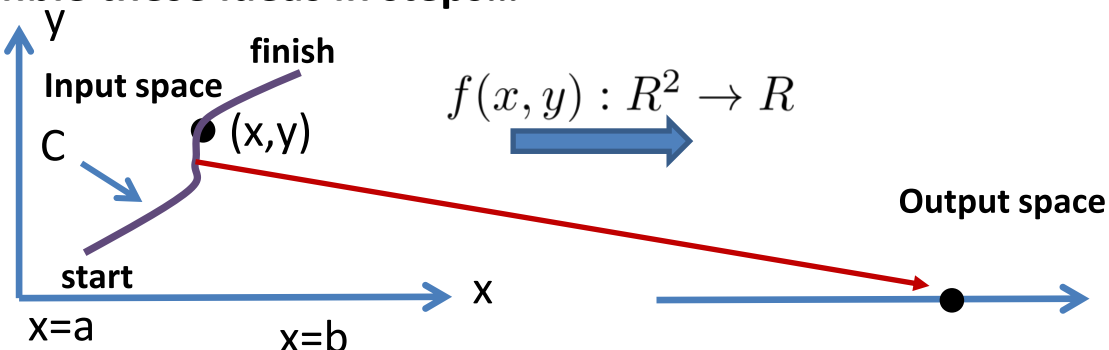
It seems clear that the $\text{Answer} = \int_a^b f(x,0)dx$ **Because y=0 along that segment**

Observation: There was *nothing special* about integrating over that line segment!

Section 16:2 We will assemble these ideas in steps…

Suppose I give you a function f(x,y) $f(x,y) : R^2 \to R$

Now, suppose I ask you to integrate f(x,y) along the purple curve C



It seems clear that the answer should be $\int_{start}^{finish} f(C)dC$

(in other words, we integrate F along little bits dC of the curve from start to finish)

- **First we need to describe the curve: let's parameterize:**

  C=(x(t), y(t)); start = x(t_{start}),y(t_{start}); finish = x(t_{end}),y(t_{end})

- **So now, we have** $\int_{t_{start}}^{t_{end}} f(x(t),y(t))dC$

- **Final question: what is dC? It's not dC = dt — because we need to scale by arc-length:**

$$dC = \sqrt{x_t^2 + y_t^2} \quad dt$$

$$\int_{t_{start}}^{t_{end}} f(x(t),y(t)) \boxed{\sqrt{x_t^2 + y_t^2}} \; dt$$

You can think of this as the "magic factor"; just like dxdy = r dr dθ

Called the "line integral" of f along C

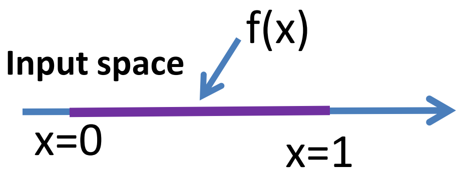
Section 16:2 We will assemble these ideas in steps…

- **Let me explain that idea again of "scaling by arc-length"**

$$dC = \sqrt{x_t^2 + y_t^2} \quad dt \qquad \int_{t_{start}}^{t_{end}} f(x(t), y(t)) \sqrt{x_t^2 + y_t^2} \quad dt$$

We'll go back to 1D to illustrate it:

Version 1: Suppose we integrate f(x) over the line segment from x=0 to x=1



$$\text{Answer} = \int_{x=0}^{x=1} f(x) dx$$

= total amount of "f" between 0 and 1

Version 2: Suppose we integrate f(x) over the line segment parameterized by x(t)=t^2, 0 < t < 1

Observe: it's the same line segment. So we should get the same total amount of "f"

dx = 2t dt

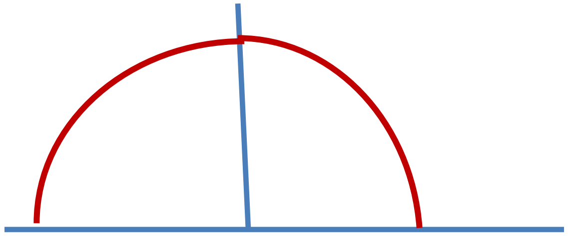
$$\text{Answer} = \int_{x=0}^{x=1} f(x) dx = \int_{t=0}^{t=1} f(x(t)) 2t dt = \int_{t=0}^{t=1} f(t^2) 2t dt$$

So, we "scaled by arc-length, which happened when we wrote dx = 2t dt

And for a curve C=(x(t),y(t)), we have $dC = \sqrt{x_t^2 + y_t^2} \quad dt$

Section 16:2 Examples $\int_{t_{start}}^{t_{end}} f(x(t),y(t))\sqrt{x_t^2+y_t^2}\ \ dt$

Find the line integral of f(x,y) = x^2 around the top half of half of the unit circle



First, we parameterize the top half of the unit circle: x(t) = cos (t), y(t) = sin(t):) 0 <t< π

Then, we evaluate the magic factor: $dC = \boxed{\sqrt{x_t^2+y_t^2}}\ \ dt$

$$x_t = -\sin t,\quad y_t = \cos t \rightarrow dC = [\sqrt{(-\sin t)^2+(\cos t)^2}]dt = 1dt$$

$$\int_{t_{start}}^{t_{end}} f(x(t),y(t))\ \boxed{\sqrt{x_t^2+y_t^2}}\,dt = \int_0^{\pi} f(\cos t, \sin t)\ \boxed{1}\ dt$$

$$= \int_0^{\pi} \cos^2 t\ \ dt = \int_0^{\pi}\ \frac{1+\cos 2t}{2}\ \ dt = \Big|_0^{\pi}\ \frac{t}{2} + \frac{1}{4}\sin 2tdt = \frac{\pi}{2}$$

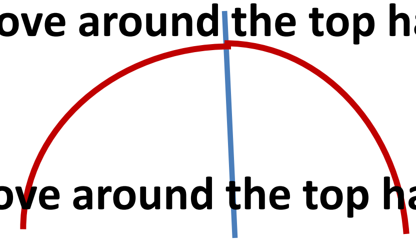
Section 16:2 More examples

We could find two more integrals, each of which will be important

(1) Find the integral of f(x,y) = x^2 with respect to x as you move around the top half of half of the unit circle:

or

(2) Find the integral of f(x,y) = x^2 with respect to y as you move around the top half of half of the unit circle



So, question (1) is asking "suppose you move around the curve C, and ask " what is total amount of F produced by the changes in x?"

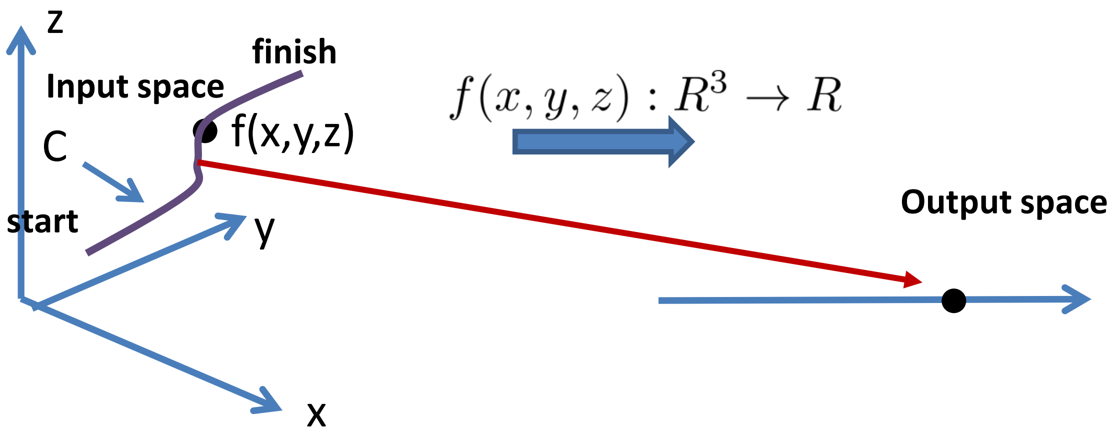
Answer to 1: $\int_C f(x,y)dx \quad = \quad \int_C f(x(t),y(t)) \boxed{\frac{dx}{dt}} dt$ Since dx = (dx/dt) * dt

Answer to 2: $\int_C f(x,y)dy \quad = \quad \int_C f(x(t),y(t)) \boxed{\frac{dy}{dt}} dt$ Since dy = (dy/dt) * dt

Section 16:2 3D

Everything I just said works in 3D….

Suppose you have a scalar function f(x,y,z):, with curve C given by C(t)=(x(t),y(t),z(t))



$$f(x,y,z) : R^3 \rightarrow R$$

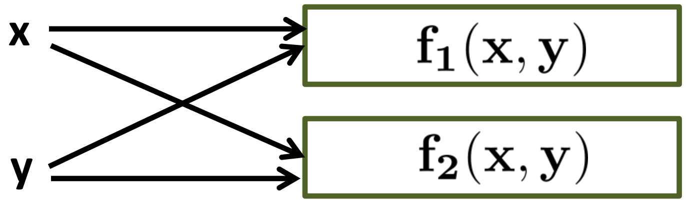
$$f(x,y,z) : R^3 \rightarrow R$$

Then the line integral of f over C is given by

$$\int_{t_{start}}^{t_{end}} f(x(t),y(t),z(t))\sqrt{x_t^2+y_t^2+z_t^2}\ \ dt$$

Section 16:2 Vector Line Integrals

Now it gets funky…. **Suppose you had a vector-valued function** $\vec{F}(x,y) = (f_1(x,y), f_2(x,y))$



x component of the vector field $\vec{F}$

y component of the vector field $\vec{F}$

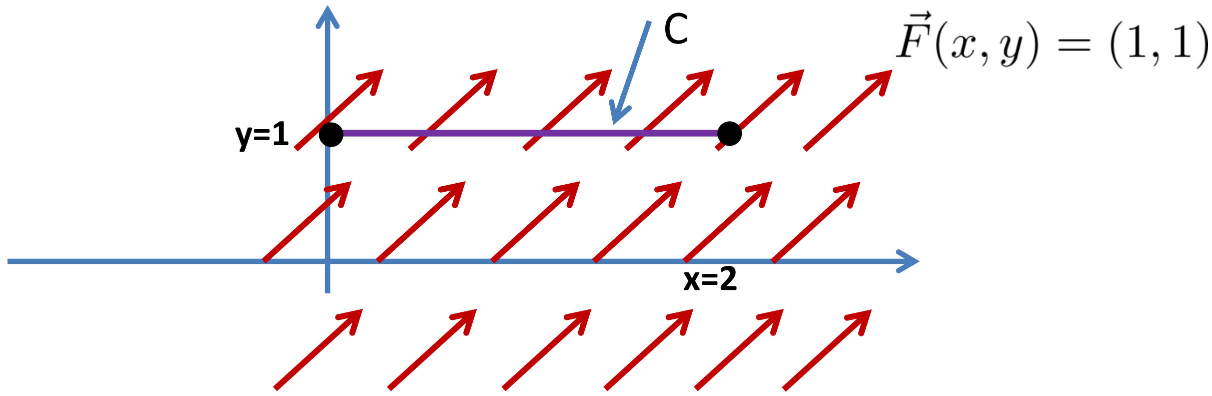
And you had a curve C passing through that vector field

What does it mean to “integrate the vector-valued F over the curve C”?

Section 16:2 Integration of a vector field over a curve C

Let's start with something simple:

Consider the vector field $\vec{F}(x,y) = (1,1)$ **and a curve C = the segment from (0,1) to (2,1)**



What does it mean to "integrate the vector-valued F over the curve C"?

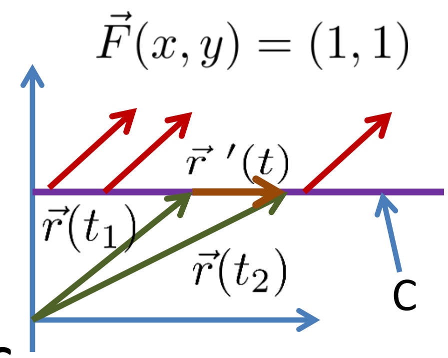
Problem: F is two-dimensional, and the curve C is one-dimensional

Solution: we can talk about the projection of F onto C

Let $\vec{r}(t)$ **be a vector that points to a spot on C**

Then $\vec{r}\,'(t)$ **is tangent to the curve C: unit tangent** $\vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

Then $\vec{F} \cdot \vec{r}\,'(t)$ **is the tangential component of** $\vec{F}$ **along the curve C**



Def: The line integral of $\vec{F}$ **along C is**

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{r}\,'\, dt = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'\, dt = \int_C \vec{F} \cdot \vec{T}$$

Section 16:2 Vector Line Integrals

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{r}\,' \, dt = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,' \, dt = \int_C \vec{F} \cdot \vec{T}$$

This is the total amount of tangential component of $\vec{F}$ along C

Example: Consider the vector field F(x,y) = (0,1)

What is $\int_C \vec{F} \cdot d\vec{r}$ if the curve C is

(a) The line segment from (0,0) to (1,0)?

$$\vec{r}\,(t) = (t, 0), \quad 0 \le t \le 1 \rightarrow \vec{r}\,'(t) = (1, 0)$$

$$\int_C \vec{F} \cdot \vec{r}\,' \, dt = \int_C (0,1) \cdot (1,0) \quad dt = 0$$

(b) The line segment from (0,0) to (0,1)?

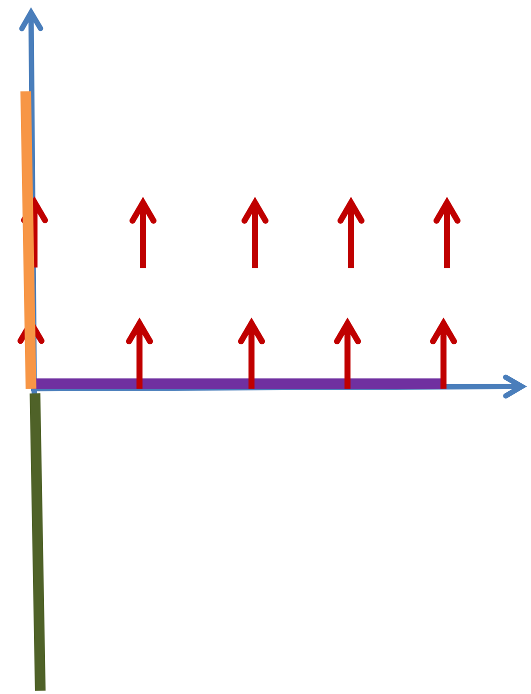
$$\vec{r}\,(t) = (0, t), \quad 0 \le t \le 1 \rightarrow \vec{r}\,'(t) = (0, 1)$$

$$\int_C \vec{F} \cdot \vec{r}\,' \, dt = \int_C (0,1) \cdot (0,1) \quad dt = 1$$

(c) The line segment from (0,0) to (0,-1)?

$$\vec{r}\,(t) = (0, -t), \quad 0 \le t \le 1 \rightarrow \vec{r}\,'(t) = (0, -1)$$

$$\int_C \vec{F} \cdot \vec{r}\,' \, dt = \int_C (0,1) \cdot (0,-1) \quad dt = -1$$



And all this makes sense, if you think about F being "work"

An extra slide: $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{r}\,' \, dt = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,' \, dt = \int_C \vec{F} \cdot \vec{T}$

Let me explain where the above expression comes from:

Along a give curve C, we want to integrate the tangential component of a vector field $\vec{F}$

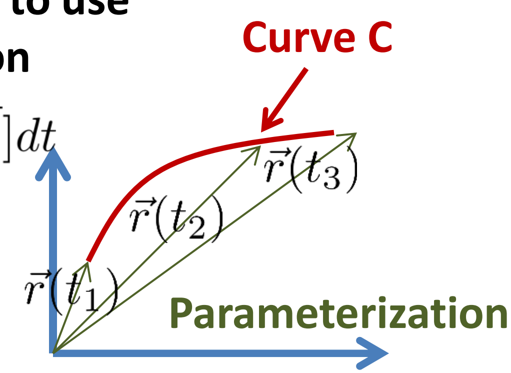
So that means we want to compute $\int_C \vec{F} \cdot T$ **, where** $\vec{F}$ **is the vector field and T is the unit tangent (that means, the unit length tangent vector to the curve C)**

Let's work backwards:

(a) we describe the curve C by the vector $\vec{r}(t)$ **, parameterized from beginning to end**

(b) we want to integrate $\vec{F} \cdot T$ **over the curve C, so we need to use arc-length in order to integrate over the parameterization**

$$\int_C \vec{F} \cdot T = \int_{tstart}^{tend} \vec{F} \cdot T \ [\vec{r}\,'(t)]dt = \int_{tstart}^{tend} \vec{F} \cdot T \ [\sqrt{x_t^2 + y_t^2}]dt$$



(c) the unit tangent vector is $\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\vec{r}\,'(t)}{\sqrt{x_t^2+y_t^2}}$

(d) substituting back in, we have

$$\int_C \vec{F} \cdot T = \int_{tstart}^{tend} \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\sqrt{x_t^2+y_t^2}} \ [\sqrt{x_t^2 + y_t^2}]dt = \int_{tstart}^{tend} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,' \, dt$$

(e) which is what we have on the top line.

Section 16:2 Integration of a vector field over a curve C

One more thing:

Def: The line integral of $\vec{F}$ along C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F}(x(t), y(t)) \cdot \vec{r}(t)\,'\, dt$$

We now show how to do line integrals in "pieces"

Step 1: Let $\vec{F} = (P(x,y), Q(x,y))$ **So P and Q are the two components of the vector field** $\vec{F}$

Step 2: We also have that $\vec{r}\,'(t) = (x\,'(t), y\,'(t))$

Step 3: So that means

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \boxed{\vec{F}(x(t), y(t))} \cdot \boxed{\vec{r}(t)\,'} dt = \int_C \boxed{(P(x,y), Q(x,y))} \cdot \boxed{(x\,'(t), y\,'(t))} dt$$

$$= \int_C P(x,y)\, x\,'(t)\, dt + \int_C Q(x,y)\, y\,'(t)\, dt$$

$$= \int_C P(x,y)\, dx + \int_C Q(x,y)\, dy$$