Section 15.10: Change of Variable

This lecture is one of those key moments in the course when all of sudden you really understand what is going on!

Let's go back to 1D calculus

u → g(u) → x

Then we can rewrite $\int_a^b f(x)dx$ **by finding the ``magic factor''**

(1) Let x = g(u)
(2) Then dx = g'(u) du

So we can write

$$\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))\, \boxed{g'(u)}\, du$$

Finally, we need to get the limits of integration correct, so if g(c)= a and g(d)=b, then

$$\int_a^b f(x)dx = \int_c^d f(g(u))\ g'(u)\ du$$

Example: Let f(x)=(2x)^3, and suppose we want $\int_2^3 (2x)^3 dx$

We can let x = g(u)= (1/2) u. Then dx = g'(u) du = ½ du, so

$$\int_2^3 (2x)^3 dx = \int_a^b f(x)dx = \int_c^d f(g(u))g'(u)\ du = \int_c^d f(\tfrac{1}{2}u)(\tfrac{1}{2})du = \frac{1}{2}\int_c^d u^3 du$$

$$= \frac{1}{2}\int_4^6 u^3 du$$

You have been doing this all along

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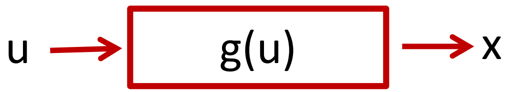
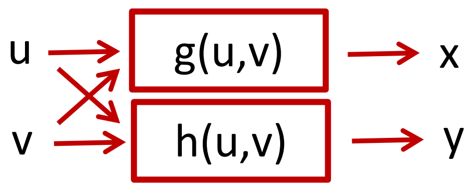
The key to changing variables was the calculation of the magic factor: dx = g'(u) du



Which allowed us to go from $\int f(x)dx \rightarrow \int g(u)[\text{magic factor}]du$

Could we find a magic factor?

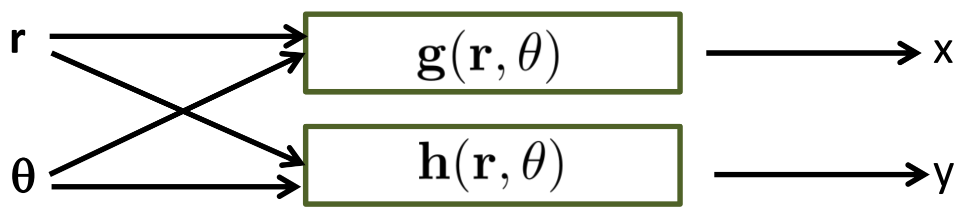
Okay, what if we had a function of two variables?



that allows us to transform variables?

$$\int\int f(x,y)dxdy \rightarrow \int\int f(g(u,v),h(u,v))[\text{magic factor}]dudv$$

Example: We have already done this for polar coordinates!!!



$$\int\int f(x,y)dxdy \rightarrow \int\int f(g(r,\theta),h(r,\theta))[\text{magic factor}]drd\theta = \int\int f(r,\theta)r\,drd\theta$$

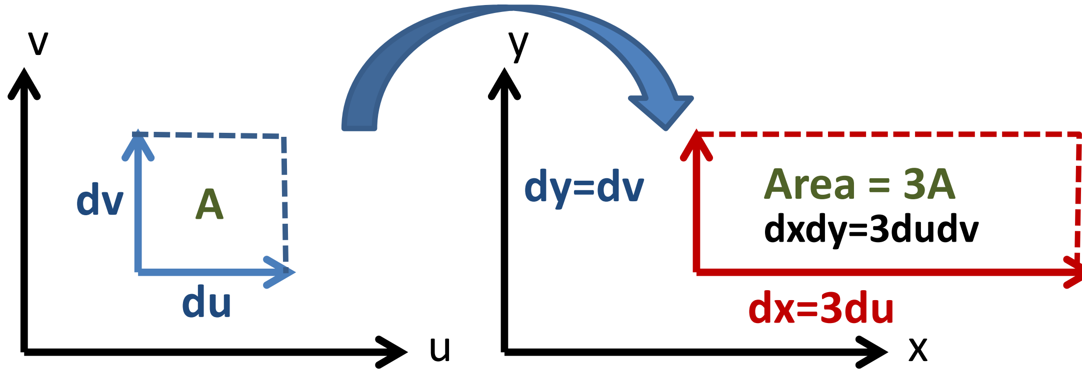
So, for polar coordinate transformations, the magic factor is **r**

How do we find it in general?

Let's build up to it:

Consider the transformation x=3u, y=v

The box dudv turns into the box dxdy

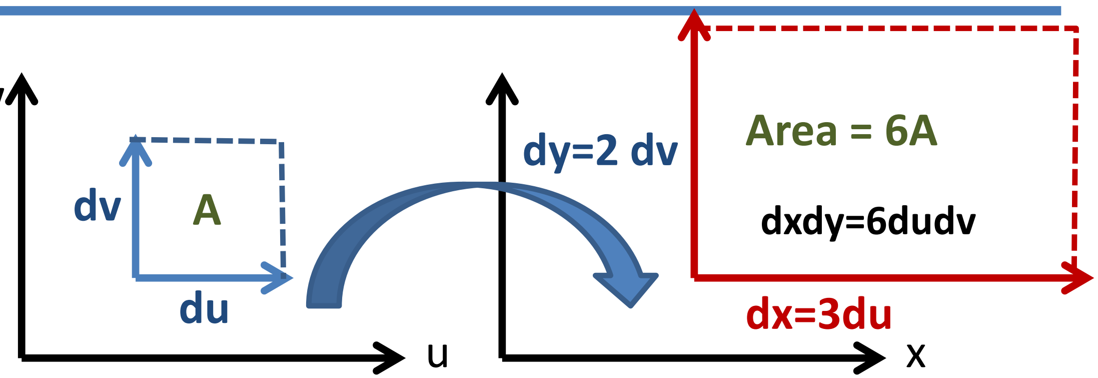


Let's take the matrix of partials

$$det \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = det \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3$$

Consider the transformation x=3u, y=2v

The box dudv turns into the box dxdy



Let's take the matrix of partials

$$det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = det \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6$$

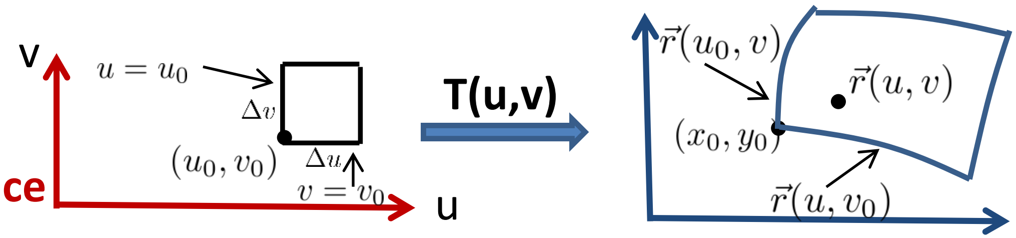
We'll prove it for a more general transformation by using linear approximations!

I want to find magic factors in general!

So, how do we find the magic factor such that ?

$$\int\int f(x,y)dxdy \rightarrow \int\int f(g(u,v),h(u,v))[\text{magic factor}]dudv$$

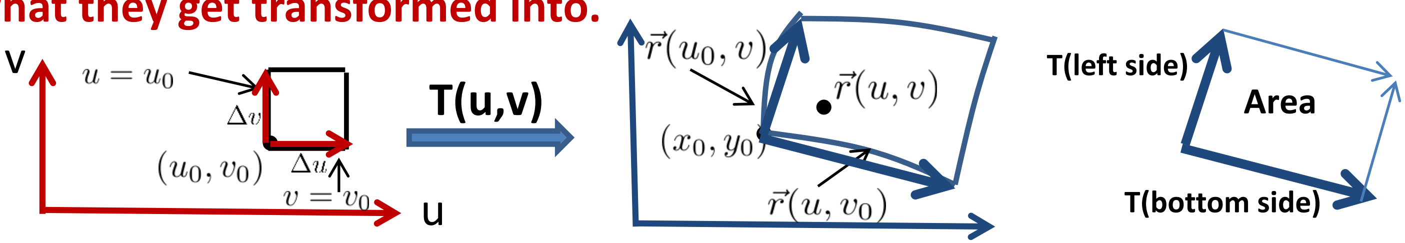
Here goes: Let's agree that $(x_0,y_0) = T(u_0,v_0)$
Standing at input (u_0,v_0), edges of the box
$\Delta u \Delta v$ get sent to different vectors in xy space



The output (x,y) comes from an input (u,v). Then any point (x,y) in output space can be written as vector $\vec{r}$ whose components $\vec{r}(u,v) = g(u,v)\vec{i} + h(u,v)\vec{j}$ depend on u and v

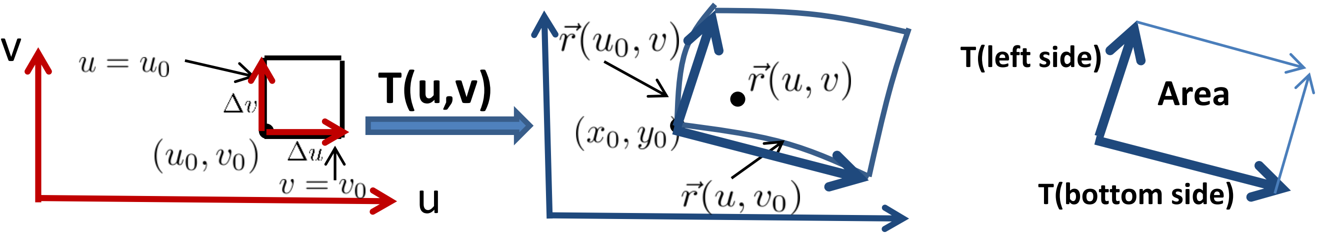
[x coordinate of vector = g(u,v) y coordinate = h(u,v)]

Our strategy will be to take the two vectors in the u,v plane making up a $\Delta u \Delta v$ box, and see what they get transformed into.



We will then take their cross-product to find the area of the transformed box—which is the magic stretch factor…

$$\text{Area} = \vec{T}(\text{bottom side}) \times \vec{T}(\text{left side})$$

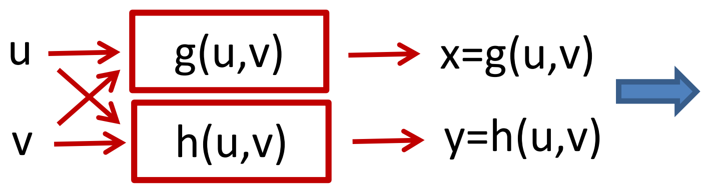


$$\vec{r}(u,v) = g(u,v)\vec{i} + h(u,v)\vec{j}$$

Step 1: Tangent vector holding v fixed, at input (u_0,v_0) is $\vec{r}_u(u,v_0) = g_u(u,v_0)\vec{i} + h_u(u,v_0)\vec{j}$

Tangent vector holding u fixed, at input (u_0,v_0) is $\vec{r}_v(u_0,v) = g_v(u_0,v)\vec{i} + h_v(u_0,v)\vec{j}$

Step 2: Remembering that



$$\vec{r}_u(u,v_0) = g_u\vec{i} + h_u\vec{j} = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j}$$

$$\vec{r}_v(u_0,v) = g_v\vec{i} + h_v\vec{j} = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j}$$

Step 3: So, now can find the vectors that make up the bottom and left sides:

bottom side vector = $\vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0) \approx \Delta u \vec{r}_u$

left side vector = $\vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0, v_0) \approx \Delta v \vec{r}_v$

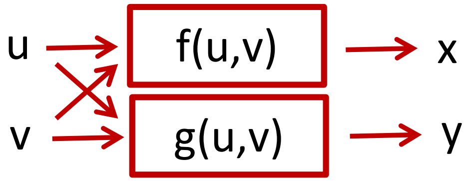
Step 4: So we can take the cross-product to find the area

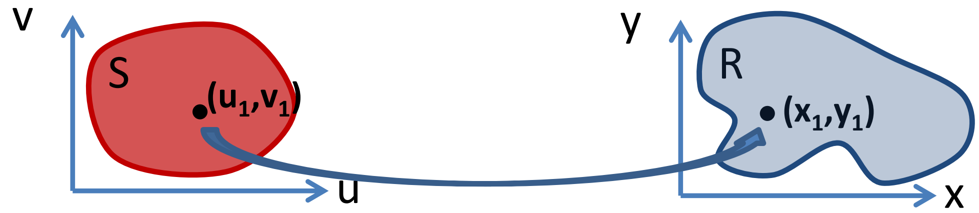
$$|(\Delta u \vec{r}_u) \times (\Delta v \vec{r}_v)| = |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v =$$

$$\left| Det \begin{bmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & 0 \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & 0 \end{bmatrix} \right| \Delta u \Delta v = \left| (0, 0, \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}) \right| \Delta u \Delta v = \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \right| \Delta u \Delta v$$

Called the Jacobian

So now have the result:



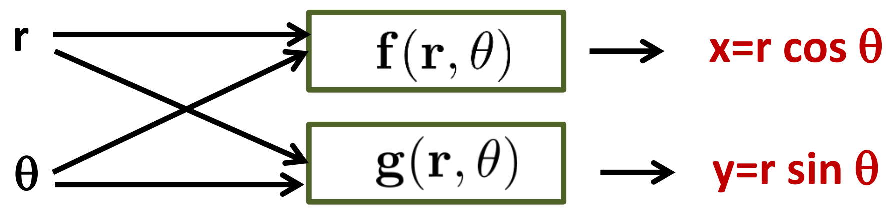


$$\int f(x,y)dxdy \rightarrow \int f(g(u,v),h(u,v))[\text{magic factor}]dudv = \int f(u,v)\left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right| dudv$$

Do you remember that for polar coordinates, we had

$$\int\int_A f(x,y)dxdy = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta$$

Let's use it right away: Prove that the Jacobian for the polar transformation
x=r cos θ, y=r sin θ is rdrdθ

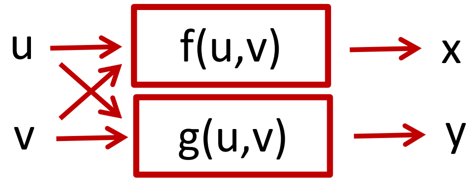


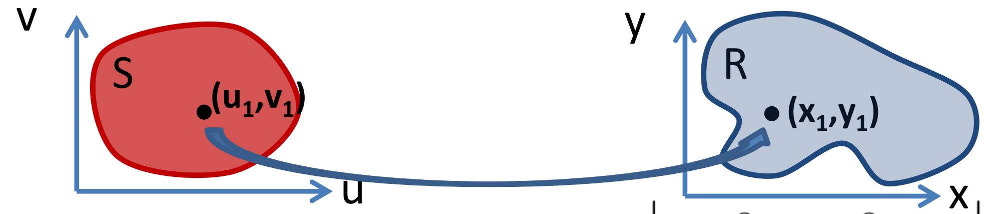
[R plays the role of u, and θ plays the role of v]

Jacobian is $= \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right| = \left|\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial y}{\partial r}\frac{\partial x}{\partial \theta}\right|$

$$= (\cos\theta)(r\cos\theta) - (\sin\theta)(-r\sin\theta) = r(\cos^2\theta + \sin^2\theta) = r$$

Let's do another example:



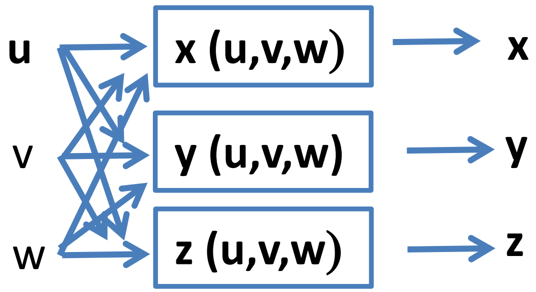


$$\int f(x,y)dxdy \to \int f(u,v)[\text{magic factor}]dudv = \int f(u,v)\left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right| dudv$$

Consider the transformation: x=uv, y=u/v Find the Jacobian of this transformation:

$$det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = det\begin{bmatrix} v & u \\ 1/v & -u/(v^2) \end{bmatrix} = (v)(-u((v^2)) - (u)(1/v) = -2u/v$$

And now you see how to find the magic factor!!!



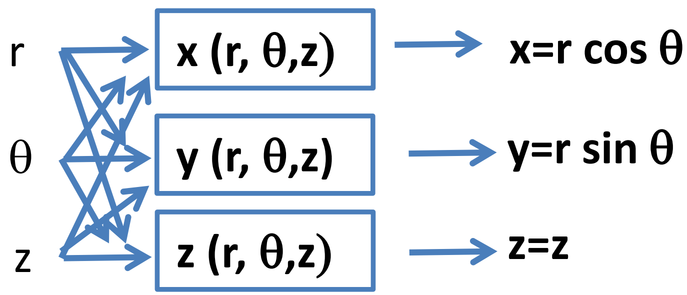
dx dy dz= [magic factor] du dv dw

Form the matrix of partials $A = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$

Then the magic factor = | det A |

Let's check it for things we know!

Cylindrical coordinates:



Geometrically, we got

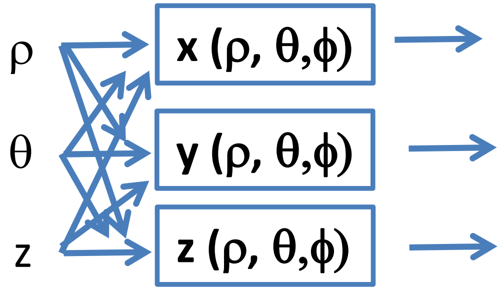
dx dy dz = r dr dθ dz

Cartesian element **Cylindrical element**

$$A = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

det A= r cos θ cos θ – (- r sin θ sin θ)
= r !!!

Spherical coordinates:



$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

And geometrically the book got **$dx\,dy\,dz = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$**

Here goes!

$$A = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\cos\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}$$

And remembering that if $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then **det A= a (ei-fh) – b (di-fg) + c (dh-eg)**

I leave it to you to check that: det A= $\rho^2 \sin\phi$!!!