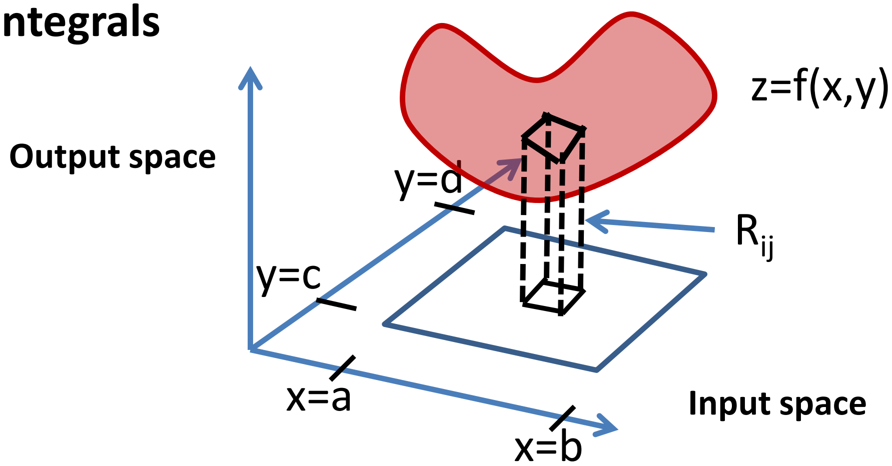
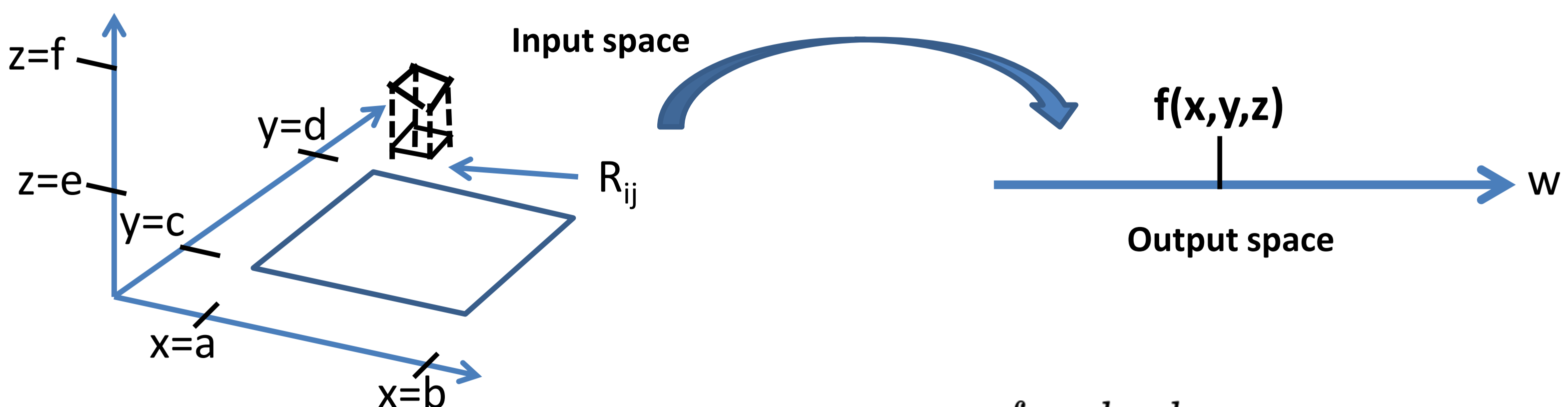
Section 15.7: Triple Integrals

So far, we have double integrals:

$$\text{Volume} = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

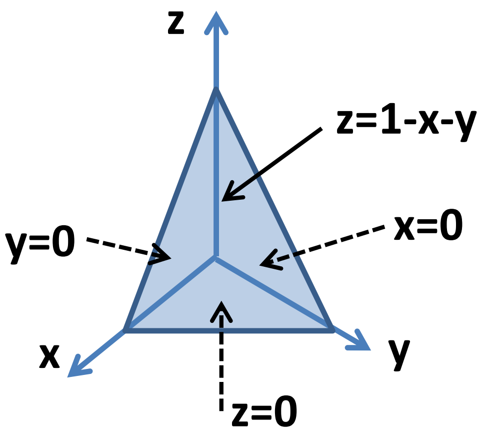


We can do the same thing for triple integrals of f(x,y,z)



$$\text{HyperVolume} = \int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Section 15.7: Triple Integrals



Okay—what about triple integrals over weird regions?

Consider the tetrahedron built from four planes:

Plane x=0, plane y=0, plane z=0, plane x+y+z=1

What is the integral of f(x,y,z)=z over this volume?

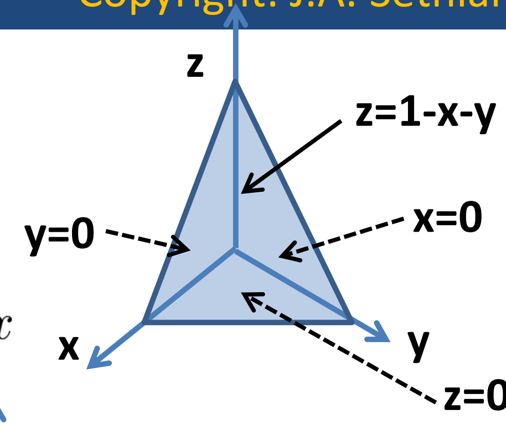
We follow the same idea:

We pick an outer region of integration (Reg:1), followed by an inner region (Reg:2) and then an even more inner region (Reg:3)

$$\int f(x,y,z)dxdydz =$$

$$\int_{Reg:1low}^{Reg:1up} \int_{Reg:2low}^{Reg:2up} \int_{Reg:3low}^{Reg:3up} f(x,y,z)d(Reg:3)d(Reg:2)d(Reg:1)$$

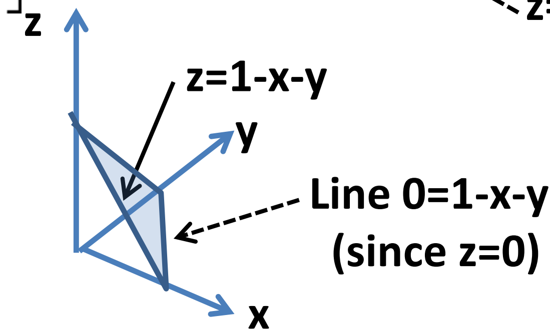
Section 15.7: Triple Integrals



And now we do the integral:

It's clear that x goes from x=0 to x=1

$$\int f(x,y,z)dxdydz = \int_0^1 \left[\int_{Reg:2low}^{Reg:2up} \int_{Reg:3low}^{Reg:3up} f(x,y,z)dzdy \right] dx$$



Step 2: For any value of x, what are the limits on y?

For any x, y starts at y=0 and ends at y=1-x

$$\int f(x,y,z)dxdydz = \int_0^1 \left[\int_0^{1-x} \left[\int_{Reg:3low}^{Reg:3up} f(x,y,z)dz \right] dy \right] dx$$

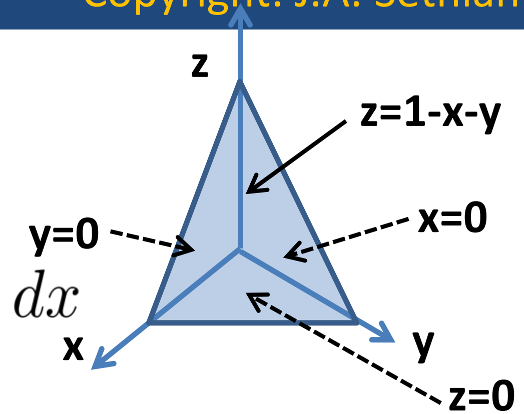
Step 3: For any value of x and y, what are the limits on z?

For any x and y, z starts at z=0 and ends at z=1-x-y

$$\int f(x,y,z)dxdydz = \int_0^1 \left[\int_0^{1-x} \left[\int_0^{1-x-y} f(x,y,z)dz \right] dy \right] dx$$

Section 15.7: Triple Integrals

Step 1: Let's pick x as the outer region



$$\int f(x,y,z)dxdydz = \int_0^1 \left[\int_0^{1-x} \left[\int_0^{1-x-y} zdz\right] dy\right] dx$$

$$= \int_0^1 \left[\int_0^{1-x} \left[\Big|_0^{1-x-y} \frac{z^2}{2}\right] dy\right] dx = \int_0^1 \left[\int_0^{1-x} \left[\frac{(1-x-y)^2}{2}\right] dy\right] dx$$

$$= \int_0^1 \left[\Big|_0^{1-x} \frac{-(1-x-y)^3}{6}\right] dx = \int_0^1 \frac{(1-x)^3}{6} dx = \frac{1}{6}\left[\Big|_0^1 - \frac{(1-x)^4}{4} = \frac{1}{24}\right.$$

Section 15.7: Triple Integrals

We still have Fubini's Theorem: If f(x,y,z) is continuous, then

$$\text{Volume} = \int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz = \int_e^f \left[\int_c^d [\int_a^b f(x,y,z)dx]dy\right] dz$$

$$= \int_e^f \left[\int_a^b [\int_c^d f(x,y,z)dy]dx\right] dz = \int_a^b \left[\int_c^d [\int_e^f f(x,y,z)dz]dy\right] dx$$

….. You get the idea…

Example: Find the integral of f(x,y,z) = x y z² over the box $0 \le x \le 1;\ -1 \le y \le 2;\ 0 \le z \le 3;$

Step 1: Set it up $\int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\,dx\,dy\,dz$

$$= \int_0^3 \int_{-1}^2 \left[\left|_0^1 \frac{x^2yz^2}{2}\right] dy\,dz = \int_0^3 \int_{-1}^2 \left[\frac{yz^2}{2}\right] dy\,dz = \int_0^3 \left[\left|_{-1}^2 \frac{y^2z^2}{4}\right] dz$$

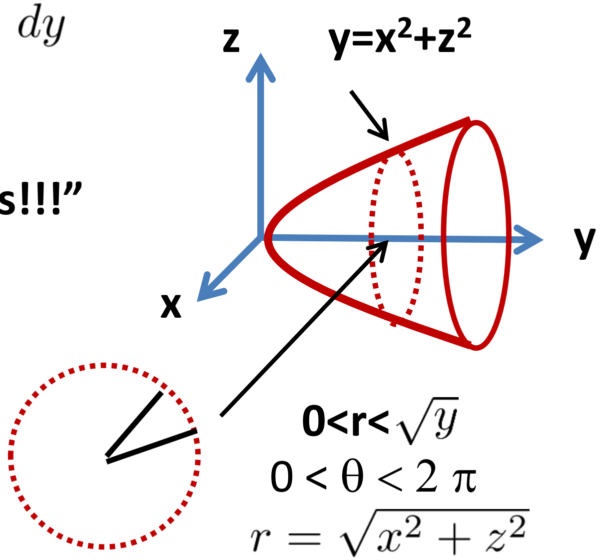
$$= \int_0^3 \left[\frac{4z^2}{4} - \frac{z^2}{4}\right] dz = \int_0^3 \left[\frac{3z^2}{4}\right] dz = \left|_0^3 \frac{3z^3}{4*3} = \frac{27}{4}\right.$$

Section 15.7: Triple Integrals

Obviously, the hard part of all this it setting up the integrals. Make pictures!!!!

Example: Evaluate the integral of $\sqrt{x^2+z^2}$ **over the region E bounded by the paraboloid** $y=x^2+z^2$ **and the plane y=4 (??????????????????)**

Idea #1: Let y go from y= 0 to y= 4 $\int_0^4 \left[\int_A \sqrt{x^2+z^2}dA\right] dy$



Idea #2: That inner integral is screaming "polar coordinates!!!"

At value y, the slice has radius $\sqrt{y}$

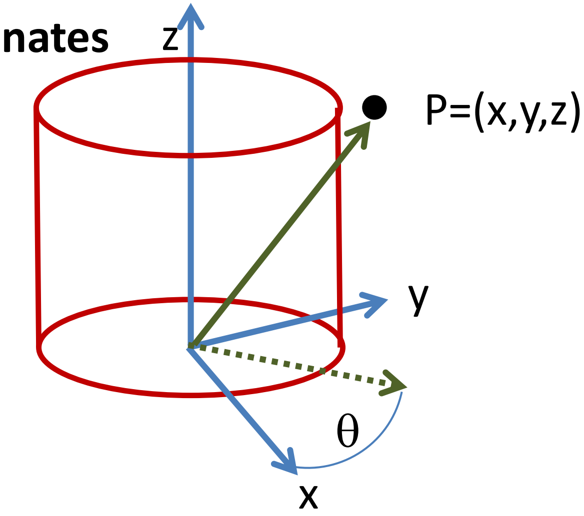
So, we have x= r cos θ , z= r sin θ so dxdz = r dr dθ and the radius r goes from 0 to $\sqrt{y}$

$$\int_0^4 \left[\int_A \sqrt{x^2+z^2}dA\right] dy = \int_0^4 \left[\int_0^{2\pi} \left[\int_0^{\sqrt{y}} r \ r dr\right] d\theta\right] dy$$

$$= \int_0^4 \left[\int_0^{2\pi} \left[\Big|_0^{\sqrt{y}} \frac{r^3}{3}\right] d\theta\right] dy = \int_0^4 \left[\int_0^{2\pi} \frac{y^{3/2}}{3} d\theta\right] dy = \int_0^4 \left[2\pi \frac{y^{3/2}}{3}\right] dy = \frac{2\pi}{3}\Big|_0^4 \frac{2}{5} y^{5/2}$$

$$= \frac{2\pi}{3}\frac{2}{5}4^{5/2}$$

Section 15.8: Triple Integrals: Cylindrical and Spherical Coordinates

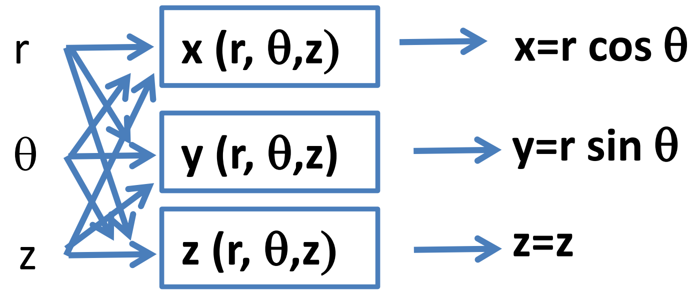


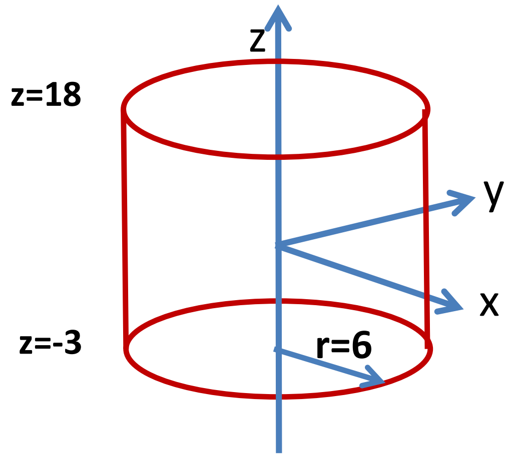
What about other 3d coordinate systems besides Cartesian?

(1) Cylindrical coordinates (r,θ,z)

r and θ are polar coordinates of the projection of the point $P=(x,y,z)$ on the x-y plane. And z is the height

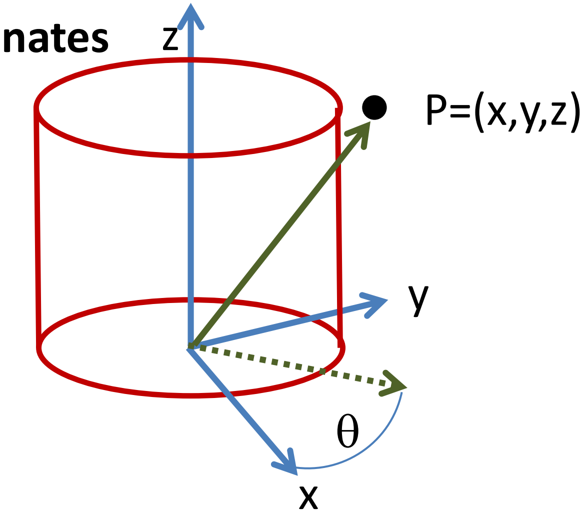
So, $x=r\cos\theta$, $y = r\sin\theta$, and $z=z$ ⇨ $r=\sqrt{x^2+y^2}, \quad z=z$





Example: $r= 6$, $-3<z<18$ is the set of all point lying on a cylinder of radius 6 with bottom at $z=-3$ and top at $z=18$

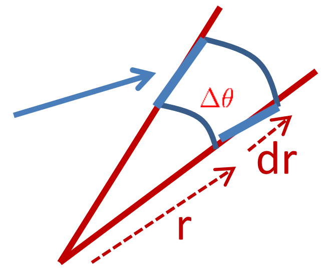
Section 15.8: Triple Integrals: Cylindrical and Spherical Coordinates



Cylindrical coordinates (r,θ, z)

Okay, what is the differential element?

Recall: $dx\,dy = r\,dr\,d\theta$



So, just extending z upwards, we must have that the differential element is

$$dx\,dy\,dz = r\,dr\,d\theta\,dz$$

Cartesian element — Cylindrical element

Example: Find the volume of a cylinder of radius 4 and height 6

(you know the answer: Volume = cross-section x height = $\pi r^2 H = \pi(16)(6) = 96\pi$

$$\text{Volume} = \int_{cylinder} f(x,y,z)dxdydz = \int_0^6\int_0^{2\pi}\int_0^4 f(x,y,z)rdrd\theta dz = \int_0^6\int_0^{2\pi}\int_0^4 1rdrd\theta dz$$

$$= \int_0^6\int_0^{2\pi} 8d\theta dz = 6 * 2\pi 8 = 96\pi$$

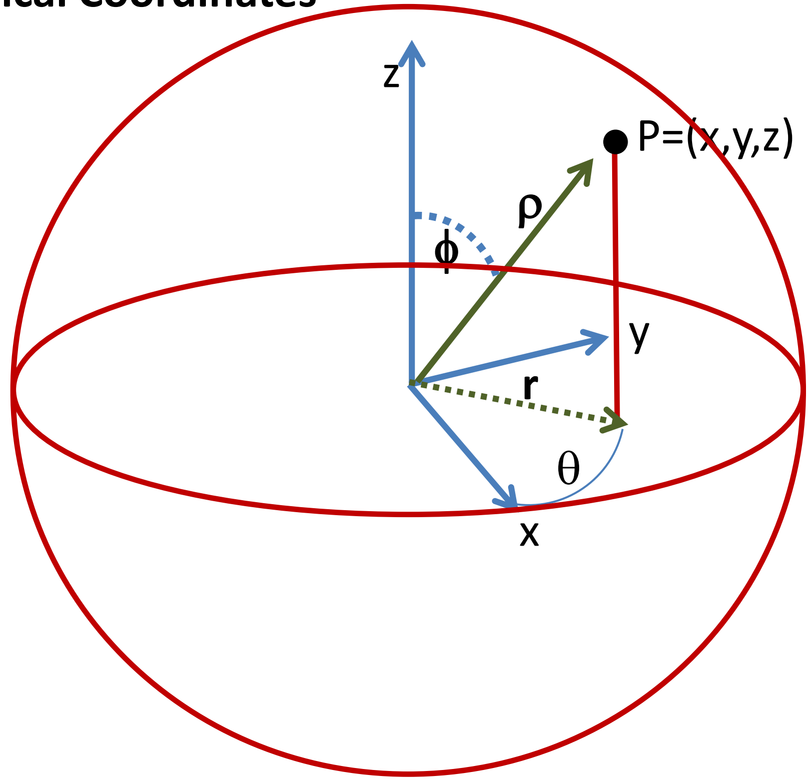
Section 15.9: Triple Integrals: Spherical Coordinates

Spherical coordinates (ρ, θ, ϕ)

$$0 \leq \rho$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \phi \leq \pi$$

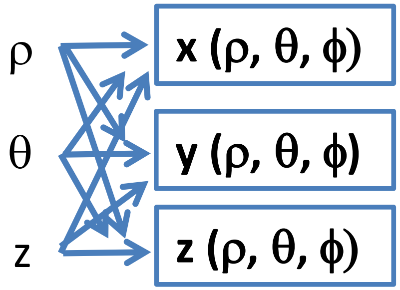


Let's find the conversion

$$x = r\cos\theta = (\rho\sin\phi)\cos\theta$$

$$y = r\sin\theta = (\rho\sin\phi)\sin\theta$$

$$z = \rho\cos\phi$$



x= (ρ sin ϕ) cos θ

y= (ρ sin ϕ) sin θ

z= ρ cos ϕ

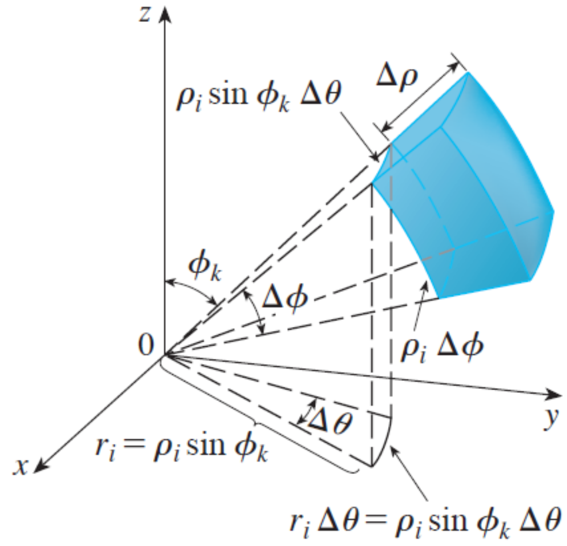
Example: What is the object $\rho=9$?

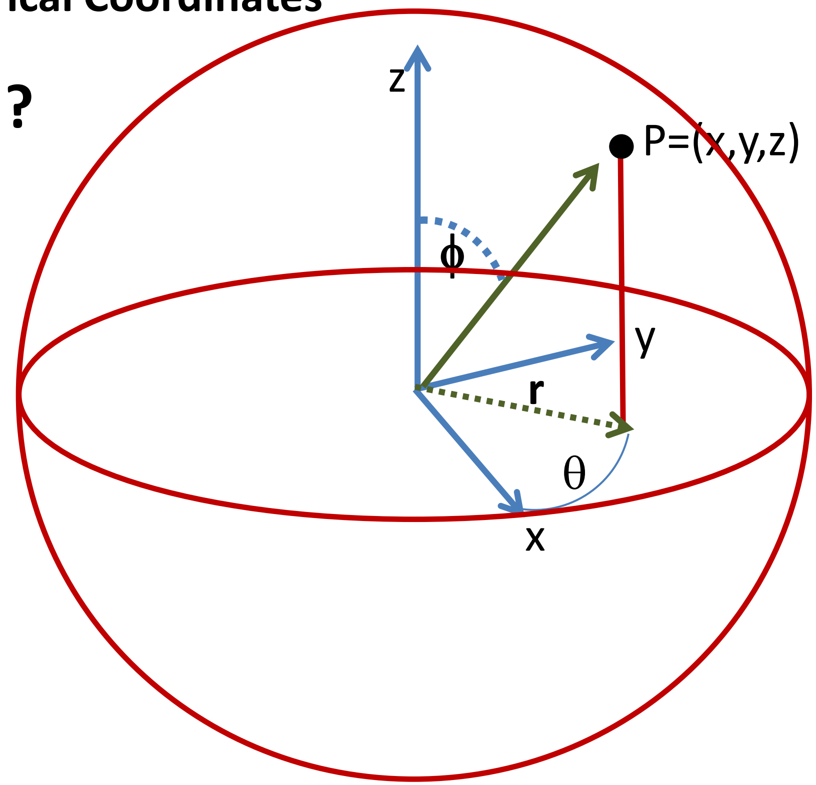
Answer: a sphere of radius 9

Section 15.9: Triple Integrals: Spherical Coordinates

So what is the differential element dx dy dz ?

Taken from Stewart





There is a long explanation in the book---I am going to tell you the answer:

$$dx\, dy\, dz = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

Please read the explanation in the book—but I'm going to give you a much neater way in the next lecture!