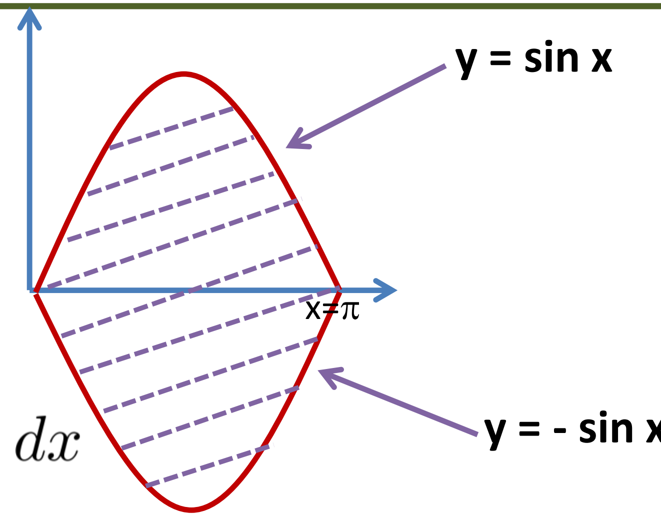
Section 15.3: To summarize

Integrate f(x,y) = 3 over the regions between y=sin x, y=-sin x, x=0 and x=π

Solution:



Step 1: Draw a picture

Step 2: Set up the integral

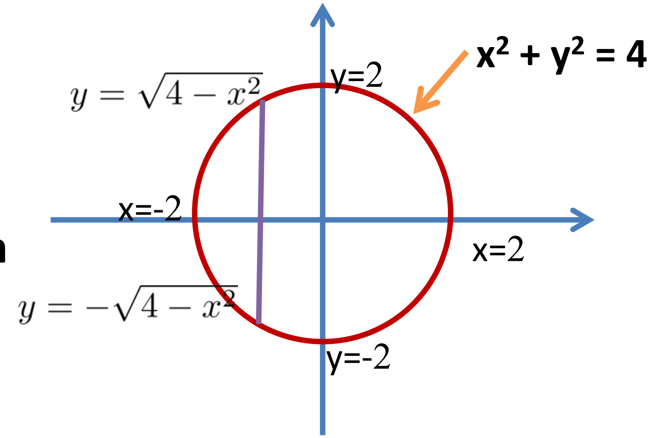
$$\int_{\text{x start}}^{\text{x end}} \left[\int_{\text{y start}}^{\text{y end}} \; dy\right] dx = \int_0^{\pi} \left[\int_{-\sin x}^{\sin x} 3 \; dy\right] dx$$

Step 3: Do the integral

$$\int_0^{\pi} \left[\int_{-\sin x}^{\sin x} 3 \; dy\right] dx = \int_0^{\pi} \left[\Big|_{-\sin x}^{\sin x} 3y\right] dx$$

$$= \int_0^{\pi} [6 \sin x] \, dx = \Big|_0^{\pi} - 6 \cos x = 6 - -6 = 12$$

What if I asked for the area inside the circle $x^2 + y^2 = 4$?



Step 1: Well, you could draw it:

Step 2: And then, given x between -2 and 2, y goes from

y start = $y = -\sqrt{4 - x^2}$

y end = $y = \sqrt{4 - x^2}$

Step 3: And then you could set up the integral:

$$\int_{\text{x start}}^{\text{x end}} \left[\int_{\text{y start}}^{\text{y end}} dy \right] dx = \int_{-2}^{2} \left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 1 \; dy \right] dx$$

Step 4: And then you could do the integral:

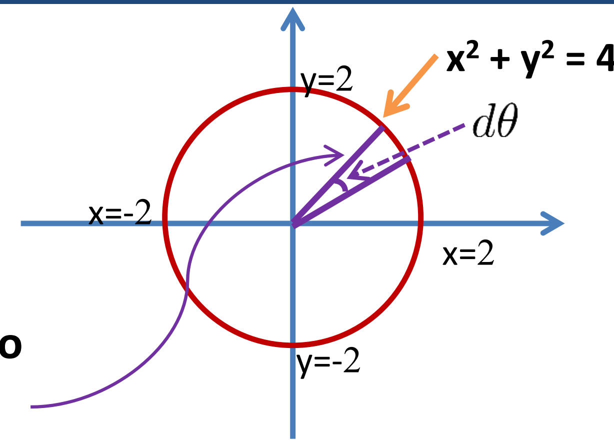
$$= \int_{-2}^{2} \left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 1 \; dy \right] dx = \int_{-2}^{2} \left[\Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y \right] dx$$

$$= \int_{-2}^{2} \left[2\sqrt{4 - x^2} \right] dx$$

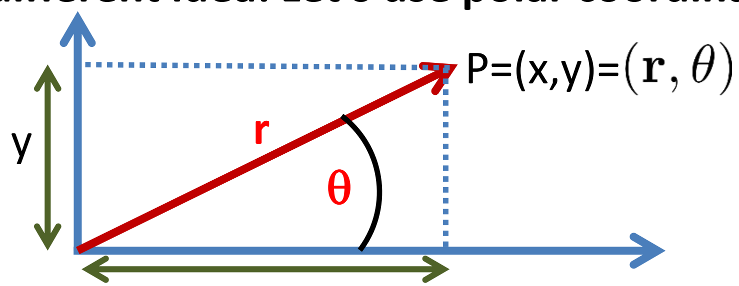
And I'm too lazy to do this integral. Any other ideas?

Section 15.4: Polar integration:

Area inside the circle $x^2 + y^2 = 4$?



A different idea: Let's use polar coordinates



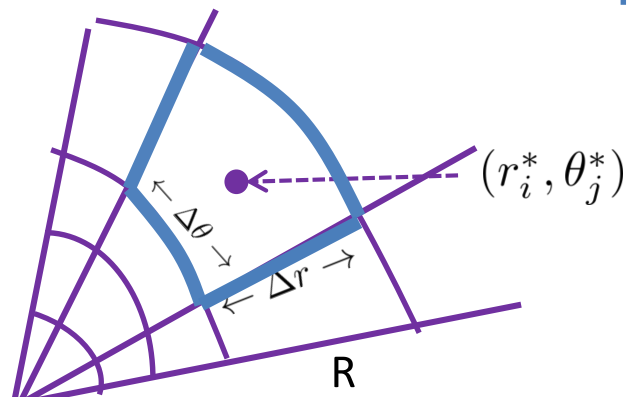
To compute the area, we want to add up "slivers"

So what is the area of each sliver?

First, we divide the sliver into pieces

(1) Let $\Delta\theta$ be the change in the angle as we sweep out the curved box

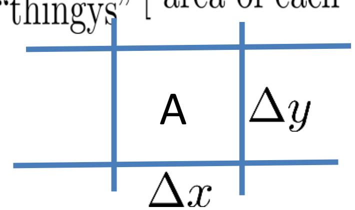
(2) Let Δr be the change in the radius as we sweep out the curved box



What is the area of this thingy?

$$Area = \sum_{\text{all "thingys"}} [\text{ area of each "thingy" }]$$

In normal Cartesian coordinates

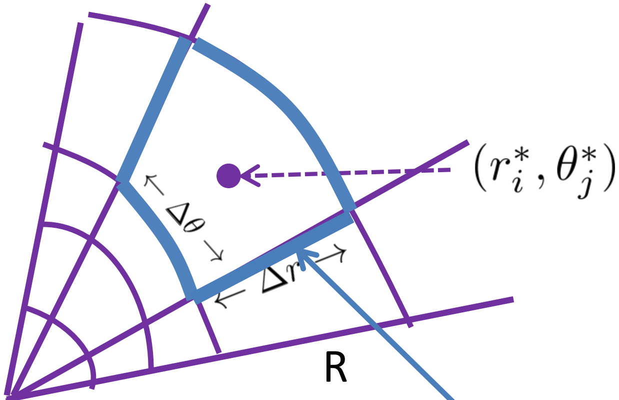


Area= $\Delta x \Delta y$

In this new coordinate system, we need to figure out the area of the curved box

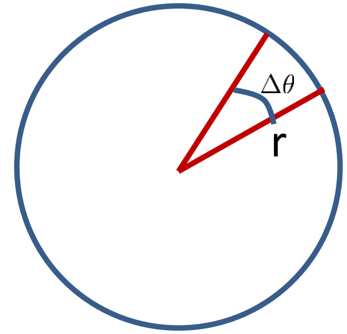
And it's not Area= $\Delta r \Delta\theta$!!!

(Now I need to explain why….)



Figuring out the area of the curved box….

Step 1: Let's remember the formula for the area of a sliver

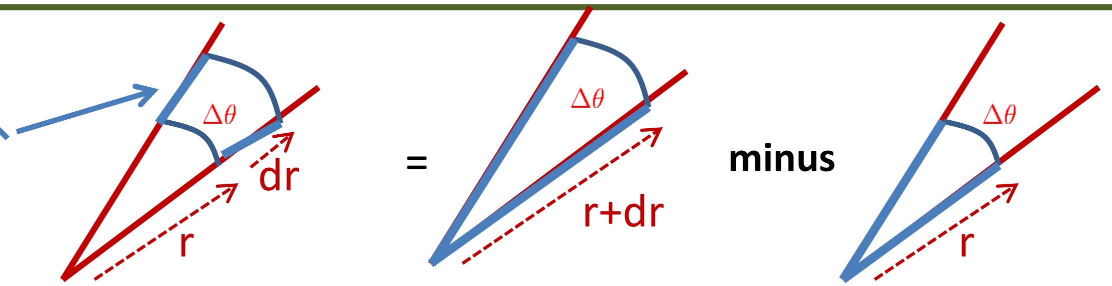


Area of whole circle = πr^2

So area of sliver is

$$\left(\frac{\Delta\theta}{2\pi}\right)\pi r^2 = \frac{r^2}{2}\Delta\theta$$

And now we can find the area of the curved box



Curved box = Whole thing - Shorter thing

$$= \frac{1}{2}(r+dr)^2 d\theta \quad - \quad \frac{1}{2}r^2 d\theta$$

$$= \frac{1}{2}(r^2 + 2rdr + (dr)^2)d\theta - \frac{1}{2}r^2 d\theta$$

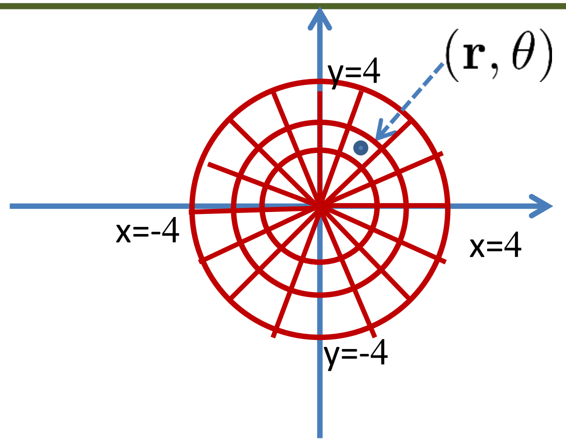
$$= \boxed{rdrd\theta} + \boxed{\frac{1}{2}(dr)^2 d\theta}$$

Term in red box goes to zero faster than term is purple box

So the area of the curved box is $AREA = rdrd\theta$

So the area of the curved box is $AREA = r dr d\theta$

Let's use it: Find the area of a circle of radius 4 using polar integration (yes, the answer is π (4)²



Step 1: We choose θ as the outer variable of integration, and r as the inner variable

$$\text{Area} = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=4} f(r,\theta) r dr d\theta$$

Step 2: The function we are integrating is just f(r,θ)=1, since it is the area

Step 3: Do the integral

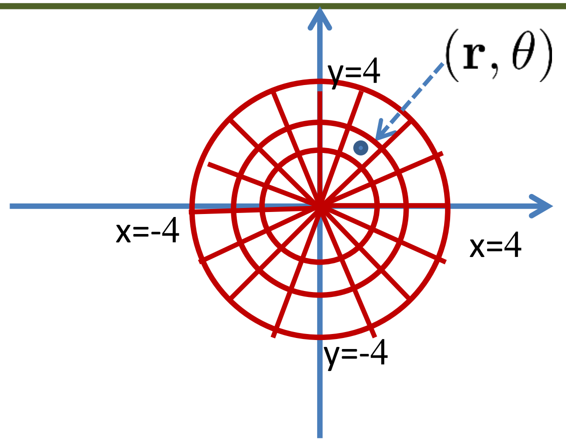
$$\text{Area} = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=4} 1 r dr d\theta = \int_{\theta=0}^{\theta=2\pi} \left[\Big|_{r=0}^{r=4} r^2/2 \right] d\theta = \int_{\theta=0}^{\theta=2\pi} 8 \quad d\theta = \Big|_{\theta=0}^{\theta=2\pi} 8\theta = 16\pi$$

So the area of the curved box is $AREA = rdrd\theta$

Suppose we did it the other way

Step 1: We choose r as the outer variable of integration, and π as the inner variable

$$\text{Area} = \int_{r=0}^{r=4} \int_{\theta=0}^{\theta=2\pi} f(r,\theta) r d\theta dr$$



Step 2: The function we are integrating is just f(r,θ)=1, since it is the area

Step 3: Do the integral

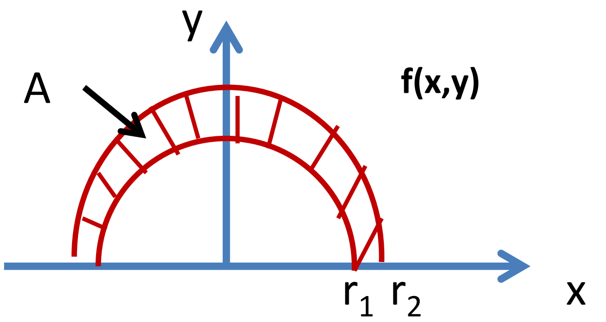
$$\text{Area} = \int_{r=0}^{r=4} \int_{\theta=0}^{\theta=2\pi} 1 r d\theta dr = \int_{r=0}^{r=4} r \left[\Big|_{\theta=0}^{\theta=2\pi} \theta \right] dr = \int_{r=0}^{r=4} 2\pi r \quad dr = \Big|_{r=0}^{r=4} 2\pi r^2/2 \qquad = 16\pi$$

(note that we can pull r out of this integral)

So the area of the curved box is $AREA = r dr d\theta$

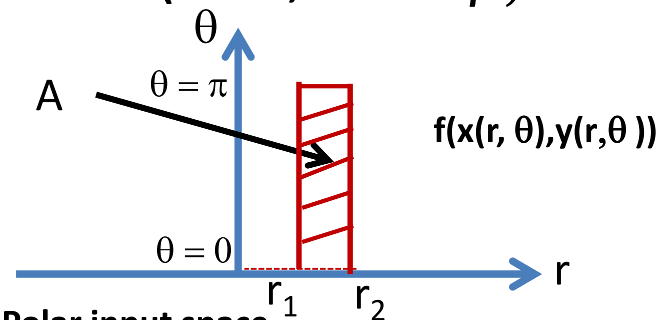
So, let's repeat:

Suppose we have a region A in xy input space, and a function f(x,y) defined in input space:



Cartesian input space

Suppose we could describe the same region in polar input space: A=(a<r<b, $\alpha < \theta < \beta$)



Polar input space

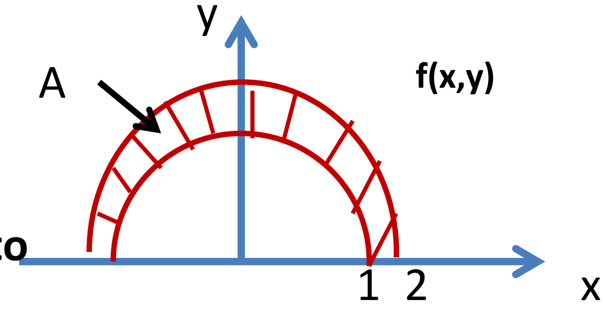
Then we can transform the integral from Cartesian coordinates to polar coordinates:

$$\int \int_A f(x,y) dx dy = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r dr d\theta$$

Let's use this ---

Example: Set up the integral to evaluate the function $f(x,y) = 3x + 4y^2$ over the region R where R is the region in the upper half plane bounded by the circles $x^2+y^2 = 1$ and $x^2+y^2=4$

Step 1: Draw the region—this is *always* the first step



So r goes from 1 to 2

Step 2: Decide which you want to be the outer integral

I choose θ, θ goes from 0 to π

Step 3: Decide which you want to be the inner integral

I choose r, r goes from 1 to 2

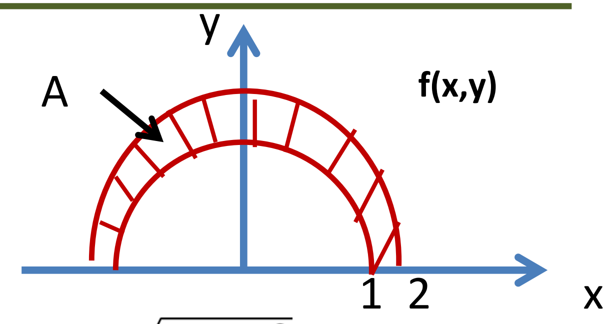
Step 4: Write down the integral

$$\int\int_A (3x+4y^2)\,dx\,dy = \int_0^\pi \left[\int_1^2 (3x+4y^2)r\,dr\right] d\theta = \int_0^\pi \left[\int_1^2 (3(r\cos\theta)+4(r\sin\theta)^2)r\,dr\right] d\theta$$

Don't forget to write x and y in terms of the polar coordinates

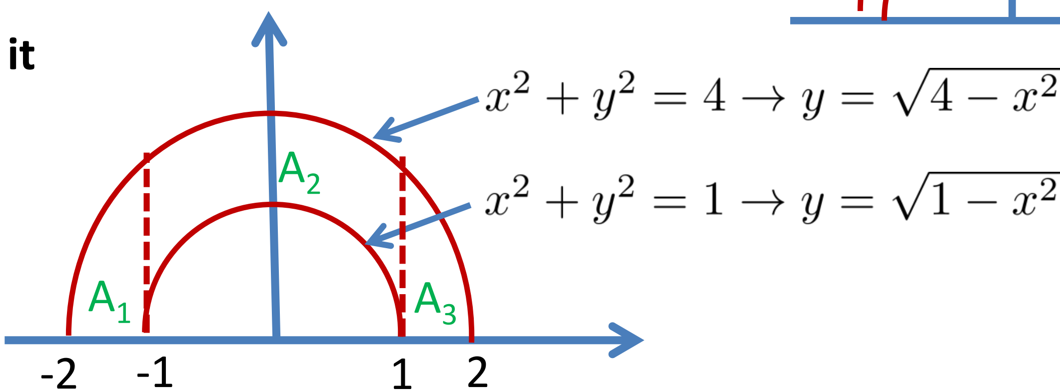
Step 5: Do the integral---I will leave this to you

$$\int\int_A (3x+4y^2)dxdy = \int_0^\pi \left[\int_1^2 (3x+4y^2)rdr\right]d\theta = \int_0^\pi \left[\int_1^2 (3(r\cos\theta)+4(r\sin\theta)^2)rdr\right]d\theta$$



I admit—this integral doesn't like fun to integrate—but look how much worse it would be in Cartesian coordinates!

We would need to break it up into three regions



$$\text{Area} = \text{Area}_1 + \text{Area}_2 + \text{Area}_3$$

$$= \int_{-2}^{-1}\int_{y=0}^{y=\sqrt{4-x^2}} (2x+4y^2)dydx + \int_{-1}^{1}\int_{y=\sqrt{1-x^2}}^{y=\sqrt{4-x^2}} (2x+4y^2)dydx +$$
$$\int_1^2\int_{y=0}^{y=\sqrt{4-x^2}} (2x+4y^2)dydx$$

And I really don't want to do this integral…..

Some final comments —be careful about direction:

θ,r increasing gives rdrdθ just like x,y increasing gives dx dy

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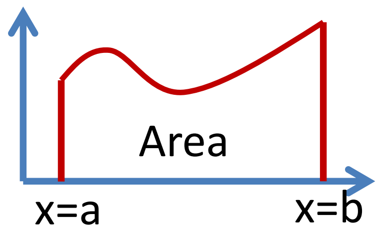
And let's make sure we understand one more thing

Question: Find the area under the curve y=f(x) between x=a and x=b

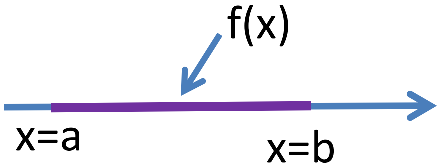
This one wants you to integrate the area under a function graphed with output against input



$$\text{Area} = \int_a^b f(x)dx$$

Question: Integrate the function f(x) along the x axis between x=a and x=b

This one wants you to integrate the total "weight" of a function describe in input space.



$$\text{Answer} = \int_a^b f(x)dx$$

Note that they are the same thing…

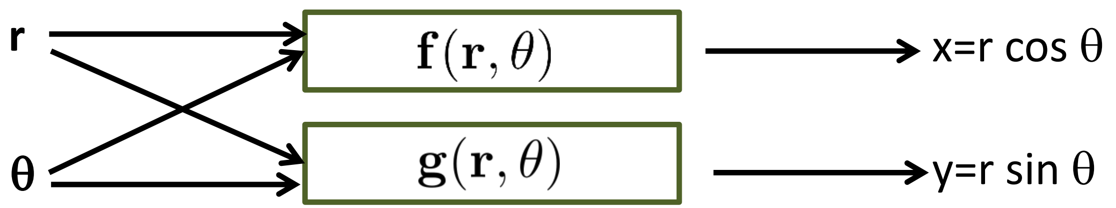
Before I go further, let me show you something amazing

Definition: A matrix is a table of numbers $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ **Math 54**

Definition: The determinant of a 2x2 matrix is det A = ad-bc

Definition: The determinant of a 3x3 matrix is det B = a (ei-fh) – b (di-fg) + c (dh-eg)

Polar coordinates



Suppose we form the "matrix of partials"

$$A = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$\det\ \mathrm{A} = (\cos\theta)(r\cos\theta) - (\sin\theta)(-r\sin\theta) = r(\cos^2\theta + \sin^2\theta) = r$$

It seems that dxdy = det A dr dθ = r dr dθ WOW!!!!!! (don't worry—I will prove this later!)

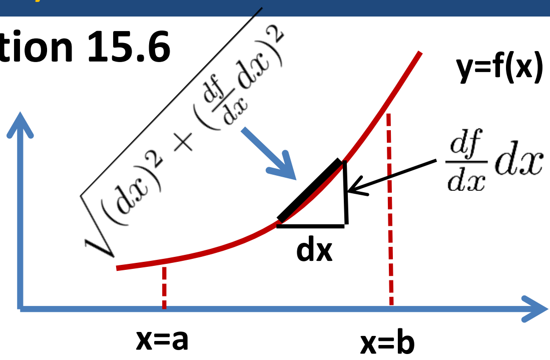
Section 15.5

Please read this section, and do the assigned homework problems….

Surface Area

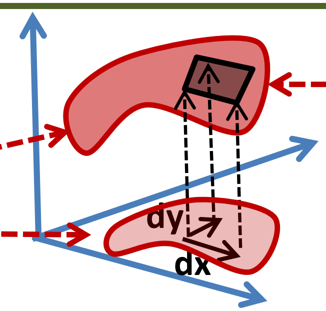
Section 15.6

In 1D: the length of the curve from (a,f(a)) to (b,f(b)) is



$$\text{Length} = \int_a^b \sqrt{1 + (\tfrac{df}{dx})^2} \;\; dx$$

What is the area of a surface z=f(x,y) that is the graph of an input region?

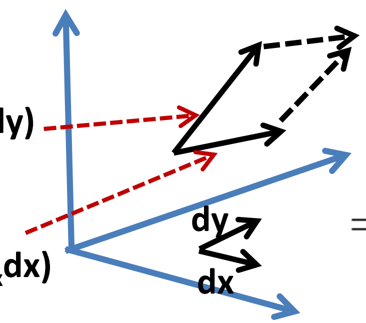


Plan: divide surface into a bunch of small surface patches

Area of patch = dSurfArea

How can we find the area of a surface patch?

Partial derivatives tell us how the sides of the patch change:



And we know the area of a parallelogram: it is

$$|\vec{u} \times \vec{v}| = |(dx, 0, f_x dx) \times (0, dy, f_y dy)|$$

$$= \begin{vmatrix} i & j & k \\ dx & 0 & f_x dx \\ 0 & dy & f_y dy \end{vmatrix} = |(f_x dxdy, -f_y dxdy, dxdy)|$$

$$= \sqrt{(f_x dxdy)^2 + (-f_y dxdy)^2 + (dxdy)^2}$$

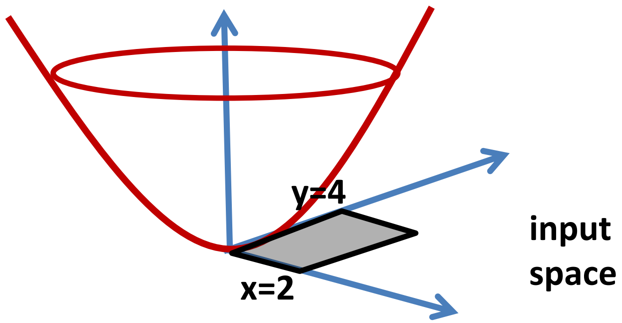
$$= \sqrt{1 + (f_x)^2 + (f_y)^2} \;\; dxdy$$

$$\int_{\substack{input \\ region}} \sqrt{1 + (f_x)^2 + (f_y)^2} \;\; dxdy$$

They are the same!

$$\int_{\frac{input}{region}} \sqrt{1+(f_x)^2+(f_y)^2}\; dxdy$$

Surface Area Example: Write down the integral representing the surface area of $z=x^2+2y^2$ above the patch $0 \le x \le 2,\ 0 \le y \le 4$



Let's use y as the outside variable

Then y goes from 0 to 4, and the inner variable x goes from 0 to 2

$$\int_0^4 \int_0^2 \sqrt{1+(f_x)^2+(f_y)^2}\; dxdy = \int_0^4 \int_0^2 \sqrt{1+(2x)^2+(4y)^2}\; dxdy$$