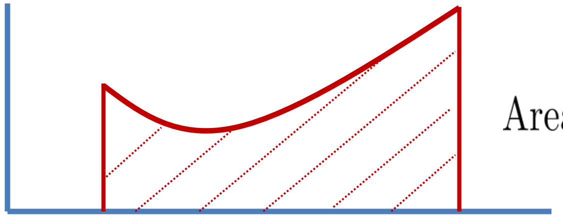
Section 15.1-15.2: On to integration!

- **We've just done differentiation**
- **Obviously, what comes next is integration (Chapter 15)**
- **And after that, Chapter 16 is the**

Fundamental theorems of multidimensional calculus

So let's begin: The good news: this looks a lot like everything you learned in 1D

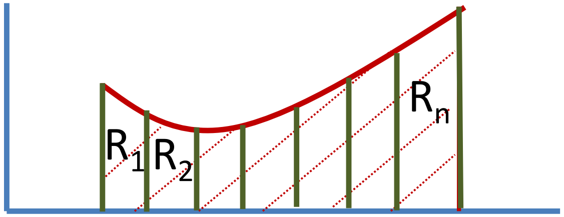
Let's recall 1D $\int_a^b f(x)dx =$ Area under the curve y=f(x) between x=a and x=b



$$\text{Area} = \int_a^b f(x)dx$$

And what is that really?

Answer: the sum of rectangles



$$\text{Area} = \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^n R_i = \lim_{\Delta x \to 0} \sum_{i=1}^n R_i$$

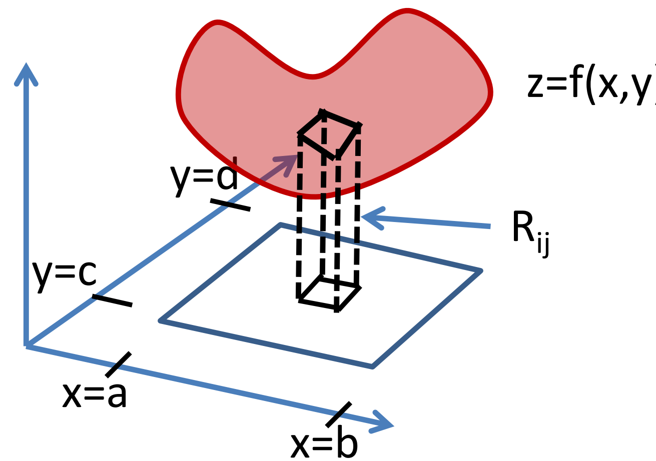
And you already knew that….

Section 15.1: On to integration!

Well, this should come as no surprise:

$$\text{Volume} = \int_c^d \int_a^b f(x,y)\,dx\,dy = \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \sum_{i=1,j=1}^{n} R_{ij}$$

Instead of summing up the area of rectangles, you sum up the volume of skyscrapers



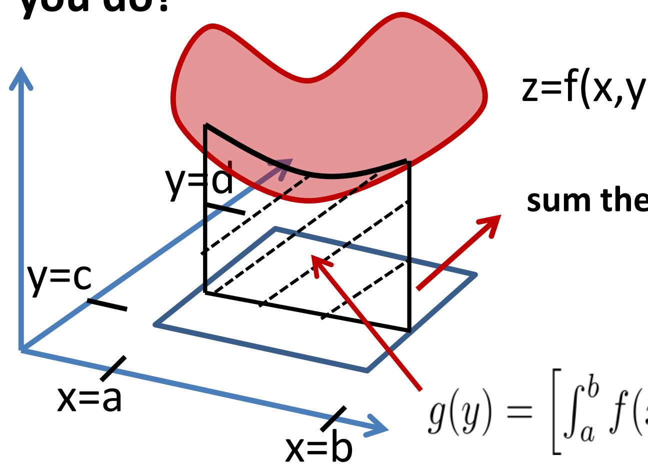
First interesting question: Does it matter what order you do?

Does $\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$?

More precisely: $\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$

This is a function of y: call it g(y)

$$= \int_c^d g(y)\,dy$$



> Fubini's Theorem: if f(x,y) is continuous , then
>
> they are equal!!! $\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$

Let's use all this. **Example: Find** $\int_0^3 \int_1^2 x^2 y dx dy$

Step 1: Rewrite with brackets

$$\int_0^3 \int_1^2 x^2 y \; dx dy = \int_0^3 \left[\int_1^2 x^2 y \; dx \right] dy$$

Step 2: Do the inner integral, holding y fixed

$$\left[\int_1^2 x^2 y \; dx \right] = \Big|_1^2 \frac{1}{3} x^3 y = \frac{1}{3}(2^3) y - \frac{1}{3}(1^3) y = \frac{1}{3}(8-1) y = \frac{7}{3} y = g(y)$$

Step 3: Put the result of the inner integral back in and finish

$$\int_0^3 \int_1^2 x^2 y \; dx dy = \int_0^3 g(y) \; dy = \int_0^3 \frac{7}{3} y \; dy = \Big|_0^3 \frac{7}{3 \cdot 2} y^2 = \frac{7}{6}(3^2 - 0^2) = \frac{63}{6}$$

Done

And because Fubini's theorem applies, let's check that it doesn't matter which way you do it

Step 1: Swap the order of integration

$$\int_1^2 \int_0^3 x^2 y \; dydx = \int_1^2 \left[\int_0^3 x^2 y \; dy\right] dx$$

Step 2: Do the inner integral, holding y fixed

$$\left[\int_0^3 x^2 y \; dy\right] = \Big|_0^3 \frac{x^2 y^2}{2} = \frac{1}{2}(3^2)x^2 - \frac{1}{2}(0^2) = \frac{9}{2}x^2 = g(x)$$

Step 3: Put the result of the inner integral back in and finish

$$\int_1^2 \int_0^3 x^2 y \; dxdy = \int_1^2 g(x) \; dx = \int_1^2 \frac{9}{2}x^2 \; dx = \Big|_1^2 \frac{9}{2 \cdot 3}x^3 = \frac{9}{2 \cdot 3}(2^3 - 1^3) = \frac{63}{6}$$

Which is the same thing!

Let's see why switching the order of integration is so valuable

Example: $\int_1^2 \int_0^\pi y\sin(xy) \quad dy \quad dx$

Realization 1: I don't want to integrate $\int_0^\pi y\sin(xy) \quad dy$

Realization 2: Let's switch and see what we get for it

$$\int_1^2 \int_0^\pi y\sin(xy) \quad dy \quad dx = \int_0^\pi \int_1^2 y\sin(xy) \quad dx \quad dy$$

$$= \int_0^\pi \left[\int_1^2 y\sin(xy) \quad dx\right] \quad dy = \int_0^\pi \left[\Big|_1^2 y(\tfrac{-1}{y}\cos xy)\right] \quad dy$$

$$= \int_0^\pi \left[-\Big|_1^2 (\cos xy)\right] \quad dy = \int_0^\pi [-\cos(2y) - -\cos y]\, dy$$

$$= \int_0^\pi [-\cos(2y) + \cos y]\, dy = \Big|_0^\pi \left[\tfrac{-1}{2}\sin(2y) + \sin y\right] = 0$$

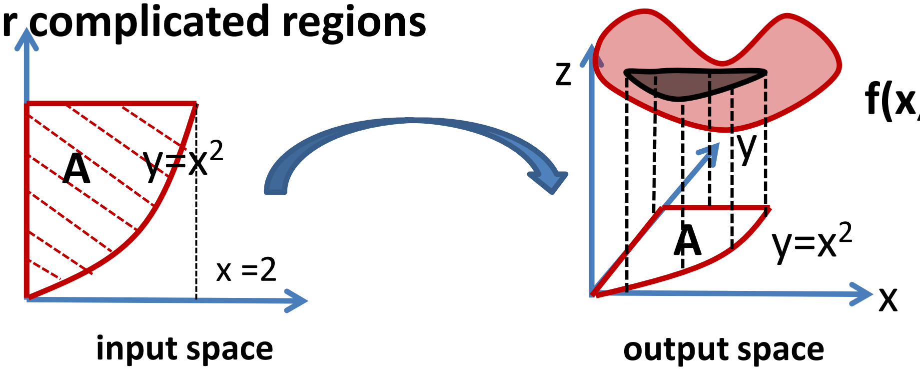
Section 15.2: Double Integrals over complicated regions

Consider $f(x,y) = x^2 y^3$

Find $\int\int_A x^2 y^3 dA$

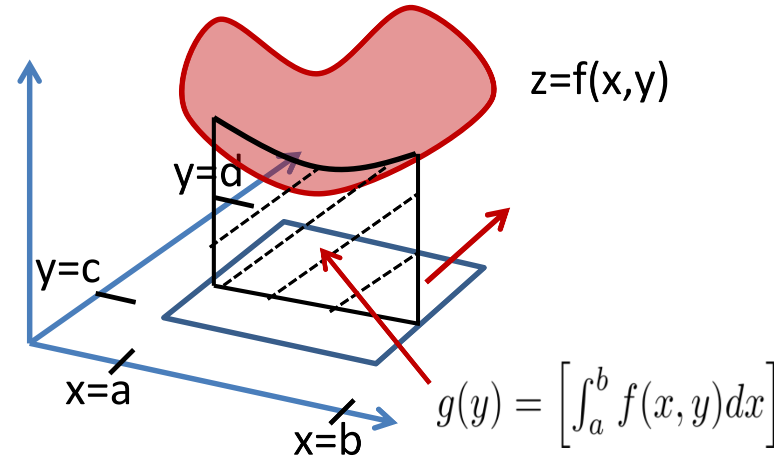
where A is the region shown in the figure

N.B. : Our answer is a volume



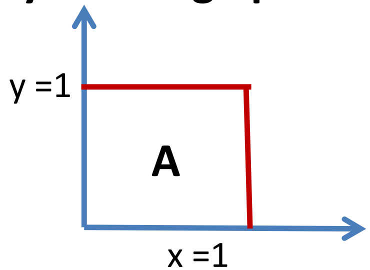
This is not as simple a problem as before—but we can do it.

The key idea will be to remember that a double integral is a sum of one-dimensional integrals—and each of those one-dimensional integrals can have different limits of integration



We will show how to do it by building up with an easier problem

Question: what is the integral of f(x,y)=2 over the unit square?

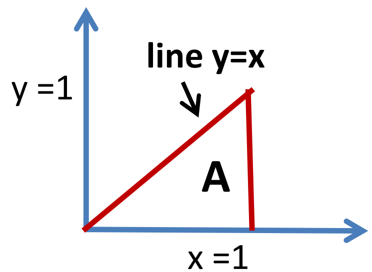


$$\int_0^1 \int_0^1 f(x,y)\ dx\ dy = \int_0^1 \int_0^1 2\ dx\ dy$$

$$= \int_0^1 \left[\Big|_0^1 2x \right] dy = \int_0^1 2dy = 2$$

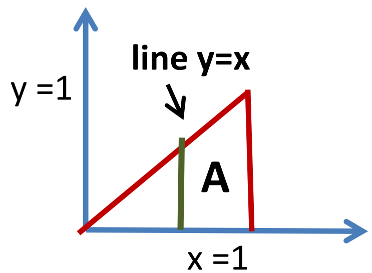
Which should be sorta obvious

Now what if I ask for the integral of f(x,y)=2 over A



Should be obvious that the answer is half of above: so the answer should be 1

But how do we set up the integral and get the limits?

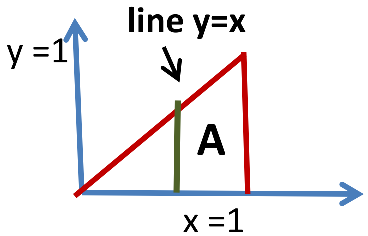


(1) x goes from 0 to 1

(2) Given x, y goes from 0 to x

(3) The contribution to the integral along the green line is $\int_0^x f(x,y)dy$

So, summing up over all the green line contributions: $\int \int_A f(x,y)dA = \int_0^1 \left(\int_0^x f(x,y)dy \right) dx$



(1) x goes from 0 to 1

(2) Given x, y goes from 0 to x

(3) The contribution to the integral along the green line is $\int_0^x f(x,y)dy$

So, summing up over all the green line contributions: $\int\int_A f(x,y)dA = \int_0^1 \left(\int_{y=0}^{y=x} f(x,y)dy\right) dx$

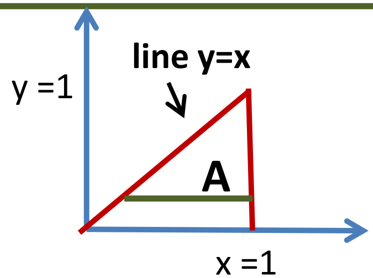
Let's do this integral and see if we get the expected value of 1

$$\int\int_A f(x,y)dA = \int_0^1 \left(\int_0^x f(x,y)dy\right) dx = \int_0^1 \left(\int_0^x 2dy\right) dx$$

Yaay!

$$= \int_0^1 \left(\Big|_0^x 2y\right) dx = \int_0^1 [2x - 2\cdot 0]\, dx = \Big|_0^1 x^2 = (1)^2 - (0)^2 = 1$$

Finally, let's do the integral the other way:



(1) y goes from 0 to 1

(2) Given y, x goes from y to 1

(3) The contribution to the integral along the green line is $\int_y^1 f(x,y)dx$

So, summing up over all the green line contributions: $\int\int_A f(x,y)dA = \int_0^1 \left(\int_y^1 f(x,y)dx\right) dy$

Check that it is the same

Yaay again!

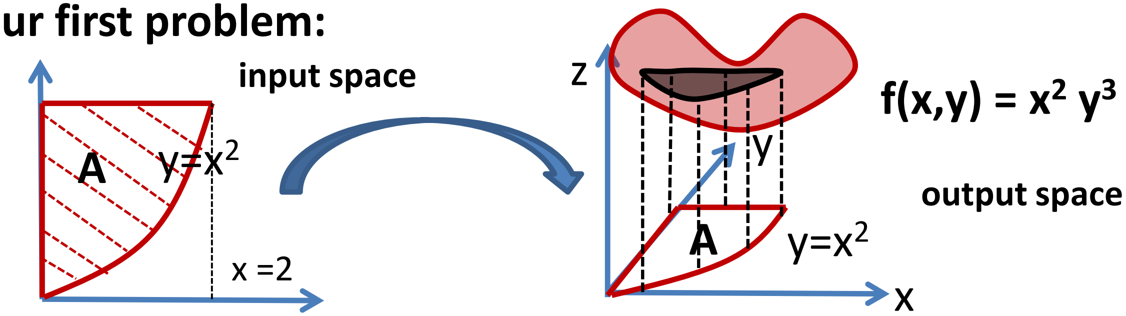
$$\int_0^1 \left(\int_y^1 f(x,y)dx\right) dy = \int_0^1 \left(\int_y^1 2dx\right) dy = \int_0^1 \left(\Big|_y^1 2x\right) dy = \int_0^1 (2-2y)\, dy = \Big|_0^1 (2y - y^2) = 2\cdot 1 - 1^2 = 1$$

Section 15.2: So let's go back to our first problem:

Consider f(x,y) = $x^2 y^3$

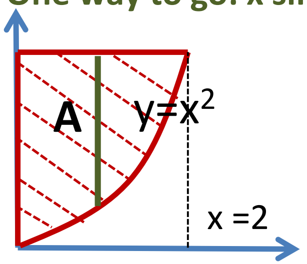
Find $\int\int_A x^2 y^3 dA$

where A is the region shown in the figure



Let's find the limits of integration: time for the green slices:

One way to go: x slices: So moving in y is the "inner integral"



inner integral

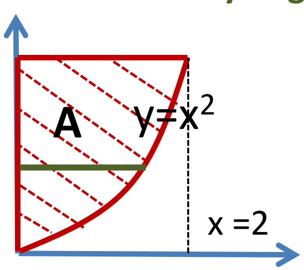
outer integral

$$\int\int_A x^2y^3dA = \int_{where\ x\ starts}^{where\ x\ ends}\int_{where\ y\ starts}^{where\ y\ ends} x^2y^3dydx$$

x goes from 0 to 2, so $= \int_0^2 \int_{\text{start of y as a fcn. of x}}^{\text{end of y as a fcn. of x}} x^2y^3dydx$

For a given x, y goes from x^2 to 4, so $= \int_0^2 \left[\int_{x^2}^4 x^2y^3\ dy\right]\ dx$

A different way to go: y slices: So moving in x is the inner integral



inner integral

outer integral

$$\int\int_A x^2y^3dA = \int_{where\ y\ starts}^{where\ y\ ends}\int_{where\ x\ starts}^{where\ x\ ends} x^2y^3dxdy$$

y goes from 0 to 4, so $= \int_0^4 \int_{\text{start of x as a fcn. of y}}^{\text{end of x as a fcn. of y}} x^2y^3dxdy$

For a given y, x goes from 0 to $\sqrt{y}$, so $= \int_0^4 \left[\int_0^{\sqrt{y}} x^2y^3\ dx\right]\ dy$

One way to go: Slicing in x: So moving in y is the inner integral

$$= \int_0^2 \left[\int_{x^2}^4 x^2y^3 \ dy\right] \ dx = \int_0^2 \left[\Big|_{x^2}^{4} \frac{x^2y^4}{4} \right] \ dx = \int_0^2 \left[\frac{x^2(4)^4}{4} - \frac{x^2(x^2)^4}{4}\right] \ dx$$

$$= \int_0^2 \left[64x^2 - \frac{x^{10}}{4}\right] \ dx = \Big|_0^2 \left[\frac{64x^3}{3} - \frac{x^{11}}{44}\right] \ dx = \frac{512}{3} - \frac{2048}{44} = \frac{16384}{132}$$

A different way to go: Slicing in y: So moving in x is the inner integral

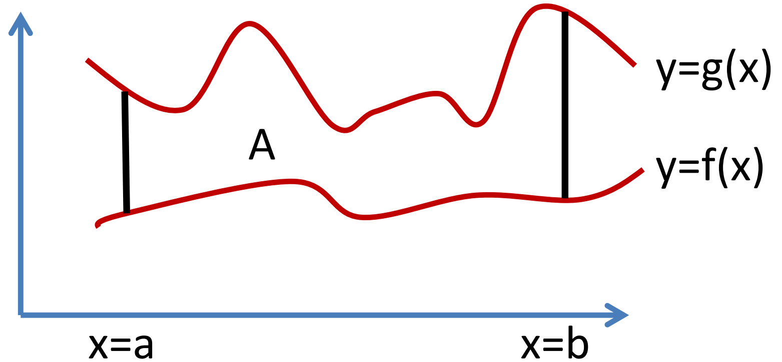
$$= \int_0^4 \left[\int_0^{\sqrt{y}} x^2y^3 \ dx\right] \ dy = \int_0^4 \left[\Big|_0^{\sqrt{y}} \frac{x^3y^3}{3} \right] \ dy = \int_0{}^4 \frac{1}{3}y^{\frac{9}{2}} dy = \Big|_0^4 \frac{2}{33}y^{\frac{11}{2}} = \frac{2}{33}2^{11}$$

$$= \frac{16384}{132}$$

Same thing

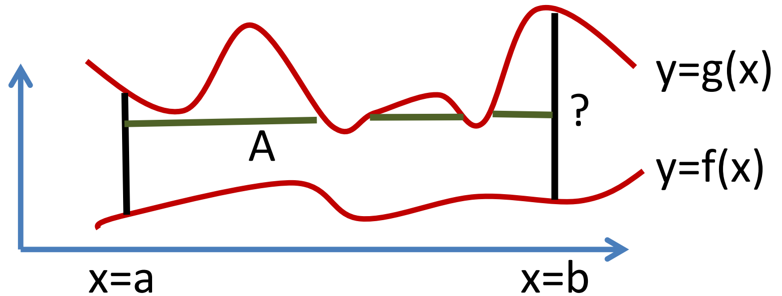
The trick with these is to think before slicing

Set up the integral of w(x,y)= sin(xy) over the region A



Inner slicing in x would be a bad idea!!!

(breaks into too many pieces)



Instead, inner slicing in y

$$\int_a^b \int_{f(x)}^{g(x)} w(x,y)dydx = \int_a^b \left[\int_{f(x)}^{g(x)} \sin xydy\right] dx$$