Question 1A: Find all locations and values of all local minima, global minima, local maxima, global maxima and saddle points of the function

$$f(x, y) = x^2 + xy + y^2 + y$$

$$f_x = 2x + y, \ f_y = x + 2y + 1$$

Set both = 0 $2x + y = 0, \ x + 2y + 1 = 0$

Solve for x and y: y=-2/3 x=1/3

Test critical point: f_{xx}= 2, f_{yy}= 2, f_{xy} = 1, so D = f_{xx} f_{yy} – $(f_{xy})^2$ = (2)(2) -1 = 3

So (1/3, -2/3) is a global and local minimum: Value is (1/3)(1/3) +(1/3)(-2/3) +(-2/3)(-2/3) + (-2/3)

Question 1B:

Let f(x,y) be the function f(x,y)=x^2 + y^2 Let D be the ellipse 4x^2 + y^2 = 4. Identify and clearly label the location of all local minimum (or minima), local maximum (or maxima), global maxima, and global minima, achieved for points (x; y) that are included in D, and give the values of f(x, y) achieved at those points.

What I wanted: f_x = 2x , f_y=2y, set = 0 equal, critical point is (0,0), f_{xx} = 2, f_{yy}=2, f_{xy}=0, D=4, so global min at (0,0), achieves a min of f(0,0)=0

Check on boundary: On the boundary given by 4x^2 + y^2 = 4,

f(x,y)= x^2 + y^2 has max at (0,2), and (0,-2), with value 4.

There is no local min, because at (1,0) and (-1,0) gives min on boundary but not over whole region.

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Question 2: True or False?

(1) Let $f(x,y)$ be a function. Let S_1 be the k_1 level set of f, S_2 be the k_2 level set of f, with $k_1 \neq k_2$. Then S_1 and S_2 cannot intersect. [TRUE]

(2) Let $\vec{u}$ be a vector in 3D. Then $|(\vec{u} \times \vec{u})| = 0$ [TRUE]

(3) For all vectors a and b, if c is the projetion of b onto a, then the projection of c onto b is zero [FALSE]

(4) Given $f(x,y)$ from R^2 to R, ∇ the direction in output space that f changes the most. [FALSE]

(5) $[\vec{u} \times \vec{v}] \times \vec{v} = \vec{u}$ [FALSE]

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Question 3:

Suppose $x^4 + 3y^2 + z^2 + xyz = 1$. Find $\frac{\partial z}{\partial x}$.

$x^4 + 3y^2 + z(x)^2 + xyz(x) = 1.$ Take partial of both sides with respect to z

$$\frac{\partial[x^4+3y^2+z(x)^2+xyz(x)]}{\partial x} = \frac{\partial 1}{\partial x}.$$

$4x^3 + 0 + 2(z)\frac{\partial z}{\partial x} + yz + xy\frac{\partial z}{\partial x} = 0$ Solve for $\frac{\partial z}{\partial x}$

$$\frac{\partial z}{\partial x} = \frac{-4x^3-yz}{2z+xy}$$

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Question 4a:

Suppose $f(x,y,z) = \sqrt{(x\log x)}ye^{xy}z$ Calculate f_{xxyyzz}

By Clairaut's theorem, we can swap the order. Taking the derivative with respect to z gets rid of all terms containing z, so doing it again to get the second partial with respect to z gives zero.

Question 4b: If $F(x,y,z,t) = x^2y^3zt^4$ **and** $x = t^2,\ y = t^3\ z = t$

Find $\frac{dF(x(t),y(t),z(t),t)}{dt}$ in terms of t

$$\frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{dF}{dt}$$

$$= (2xy^3zt^4)(2t) + (x^2 3y^2 zt^4)(3t^2) + (x^2y^3t^4)(1) + x^2y^3z4t^3$$

And now go substitute x, y and z in terms of t

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Question 5: Suppose $z = xy + f(x^2 - y^2)$ **where f is a differentiable function.**

Let $g(x,y) = y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$

Describe, parameterized by t, the level set corresponding to g=4

$$z_x = y + (f')(2x) \qquad z_y = x + (f')(-2y)$$

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = y^2 + y[(f')(2x)] + x^2 + x(f')(-2y)] = x^2 + y^2$$

So, we need to find the level set g=4, which is the circle of radius 2, so

So x = 2cos t, y = 2 sin t

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Question 6:

Find the tangent plane to $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

$$f_x = 4x \qquad f_y = 2y$$

$$(z - z_0) = f_x(x - x_0) + f_y(y - y_0) = (z - 3) = 4(1)(x - 1) + 2(1)(y - 1)$$

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Question 7a:

Consider the function $w(x, y, z) = x^2 + 3y^2 + 6z^2$.
This is a mapping from R^3 to R^1.
Let S be the level surface corresponding to $w = 24$ and let $\Gamma(t)$ be the curve on S with $x = 0$
Find $\frac{dw(\Gamma(t))}{dt}$

Since this curve is on a level surface, then w does not change along this curve. So the answer is zero.

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Question 8:

At what point (or points) on the ellipsoid $x^2 + y^2 + 2z^2 = 1$ is the tangent plane parallel to the plane $x + 2y + z = 1$.

Normal to ellipsoid is (2x,2y,4z)
Normal to plane is (1,2,1)

So need x,y,z such that (2x,2y,4z) = k (1,2,1)

So x=k/2, y=k, z=k/4

Plugging into ellipse, we have (k/2)^2 + k^2 + 2 (k/4)^2= k^2/4 + k^2 + k^/8 = 11/8 k^2 = 1

k= plus or minus square root (8/11)

Put that it to get the two points

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Question 9:

Who is the most annoying celebrity out there right now?

Lots of contenders--

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Question 10: Show that every plane that is tangent to the cone $x^2+y^2 = z^2$ passes through the origin.

Let $P=(x_0,y_0,z_0)$ be a point on the cone
Let $w(x,y,z) = z^2 - (x^2+y^2)$, so the cone is $w(x,y,z)=0$
Then the normal is $(-2x,-2y,2z)$
So the tangent plane at (x_0,y_0,z_0) is

$$(w_x*(x-x_0) + w_y *(y-y_0) + w_z (z-z_0) = 0$$

$$(-2x_0)(x-x_0) + (-2y_0)*(y-y_0) + (2*z_0)(z-z_0) = 0$$

Need to verify that the origin solves this.

Let's check let $(x,y,z)=(0,0,0)$ so this becomes

$$-2x_0^2 + -2y_0^2 + 2(z_0)^2 = 0$$

Which is true, since that's the original equation for the cone.

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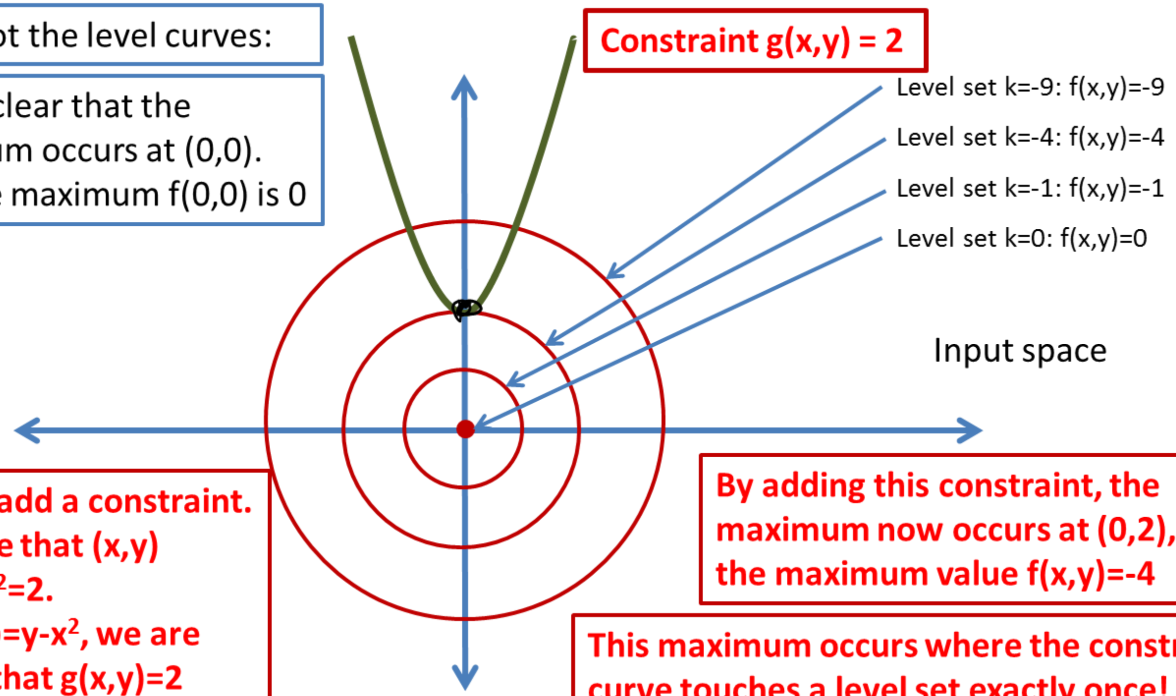
Okay---a review of Lagrange Multipliers and some problems:

Suppose we want to find the extreme points of a function f(x,y,z) subject to the constraint g(x,y,z)=k. Then at an extreme point we must have that they point in the same direction, so we must have that

Want to maximize f(x,y) = - [x^2 +y^2]

Let's plot the level curves:

Seems clear that the maximum occurs at (0,0). And the maximum f(0,0) is 0



Let's now add a constraint. We require that (x,y) satisfy $y-x^2=2$. So if $g(x,y)=y-x^2$, we are requiring that g(x,y)=2

By adding this constraint, the maximum now occurs at (0,2), and the maximum value f(x,y)=-4

This maximum occurs where the constraint curve touches a level set exactly once!

Extreme value occurs where normal to the constraint curve points in the direction as the normal to the level set

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More examples: Use Lagrange multipliers to find the maximum, and minimum values of f $f(x,y) = x^2y$ subject to the constraint $x^2+y^2=1$

$$f(x,y) = x^2y \rightarrow (f_x, f_y) = (2xy, x^2) \quad g(x,y) = x^2+y^2 \rightarrow (g_x, g_y) = (2x, 2y)$$

$$(f_x, f_y) = \lambda(g_x, g_y) \rightarrow$$

$$f_x = \lambda g_x \rightarrow 2xy = \lambda 2x \rightarrow y = \lambda \text{ or } x = 0 \rightarrow y = \lambda$$

$$f_y = \lambda g_y \rightarrow x^2 = \lambda 2y = 2\lambda^2$$

$$\text{Substitute into constraint } x^2+y^2 = 1 \rightarrow 2\lambda^2 + \lambda^2 = 1 \rightarrow 3\lambda^2 = 1$$

$$\text{So } \lambda = \pm\frac{1}{\sqrt{3}} \rightarrow y = \pm\frac{1}{\sqrt{3}} \text{ and } x = \pm\sqrt{\frac{2}{3}}$$

So points are $(\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}), (-\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}), (\sqrt{\frac{2}{3}}, \frac{1}{-\sqrt{3}}), (-\sqrt{\frac{2}{3}}, \frac{1}{-\sqrt{3}}),$

$$\text{So max is } x^2y = [\tfrac{2}{3}][\tfrac{1}{\sqrt{3}}] \quad \text{And min is } x^2y = [\tfrac{2}{3}][\tfrac{-1}{\sqrt{3}}]$$

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