Section 14.8: Lagrange Multipliers

This is very interesting—and quite geometric. Looking carefully at the drawings will really help.

Motivation: Suppose we want to minimize $f(x,y)= x^2 + y^2$ subject to the constraint $x+y = 5$

The "old way":

Step 1: In the constraint, solve for one of the variables: $y = 5-x$

Step 2: Substitute into the function to be maximized/minimized: $x^2 + y^2 = x^2 + (5-x)^2$

Step 3: Take the derivative, find where it equals zero, check second derivative:

$$f(x) = x^2 + (5-x)^2 = x^2 + 25 -10x + x^2$$

$$f'(x) = 4x -10: f'(x) =0 \text{ when } x=5/2: f''(x) = 4, \text{ so } x=5/2 \text{ is a minimum.}$$

The problem with is approach

(a) you gotta solve for one of the variables.

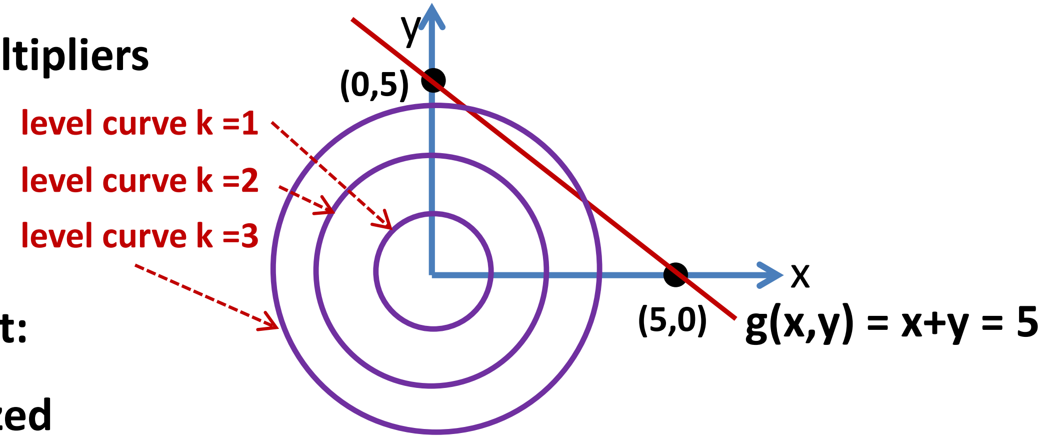
(b) when you get to worse stuff (minimize $f(x,y,z) = x^2 + 4y^2 + z^3$) it gets ugly fast!

WE NEED A BETTER WAY---HERE IT COMES!

Section 14.8: Lagrange Multipliers

Suppose we want to minimize $f(x,y)= x^2 + y^2$ subject to the constraint $g(x,y) = x+y = 5$

Let's draw input space and the constraint:



Now, let's add the function to be minimized

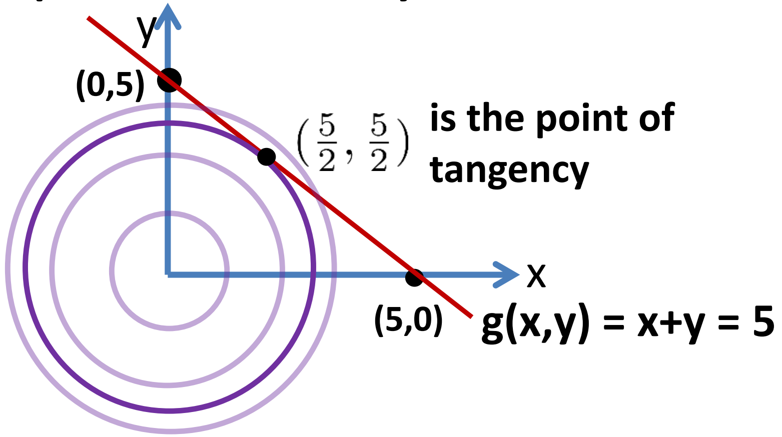
Remember: level curve k is the set of all input points sent to output f(x,y) = k

Observation #1: It seems clear that for small k, we cannot "reach" the constraint equation

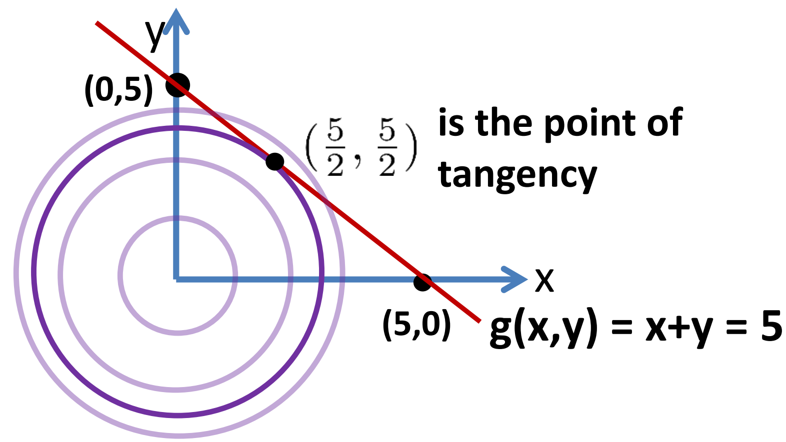
Observation #2: For larger k, we hit the constraint twice.

Remember that the goal is to minimize f(x,y) subject to the constraint, which means "find the inputs that generate the smallest output and also satisfy the constraint.

Question: What is the smallest value of k that still touches the constraint?



Claim the minimum happens at the point of tangency



Claim: if you can find a point in input space where the tangent to the level curve matches the tangent to the constraint, then you have found an optimum!

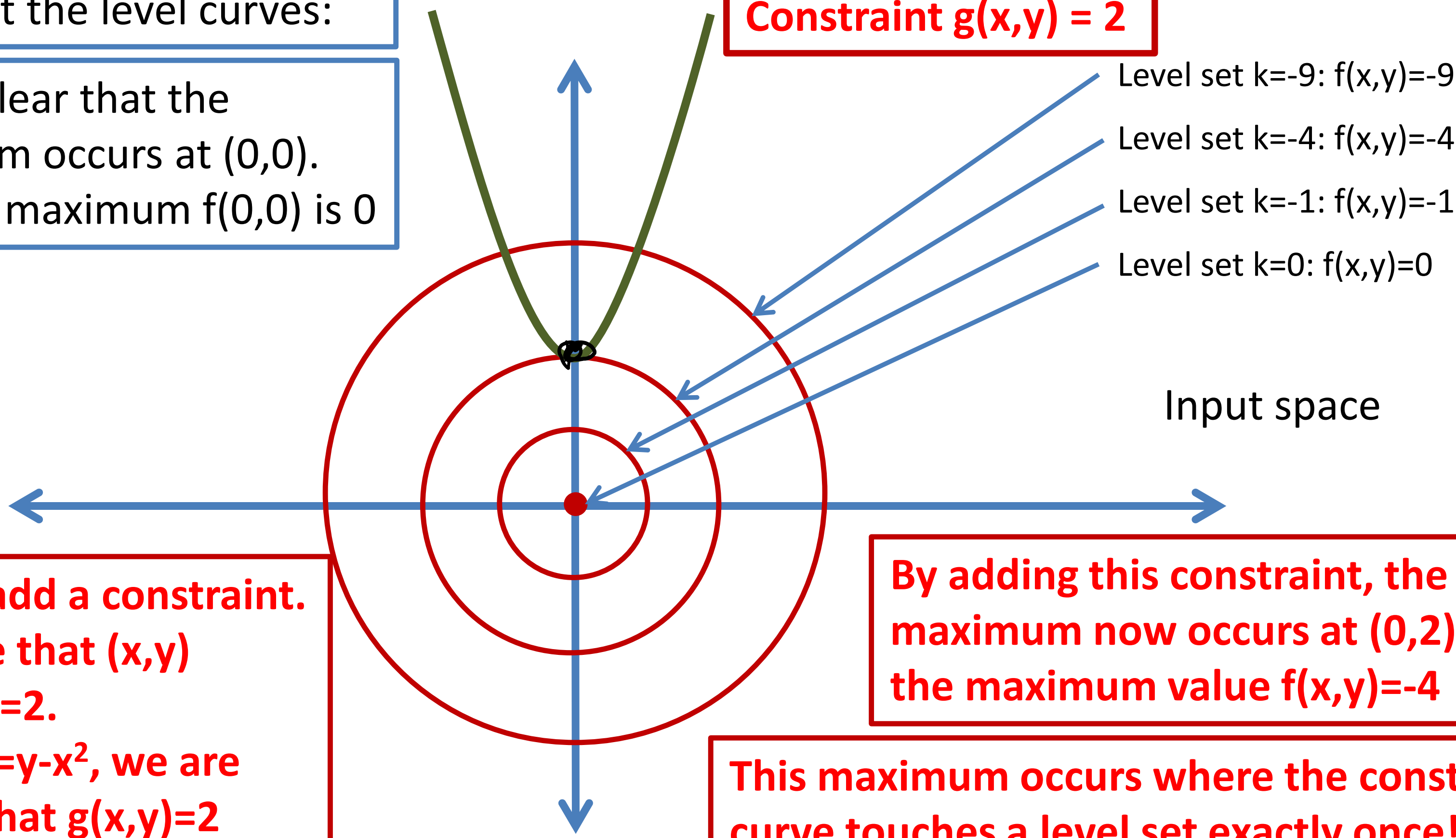
Why? Well---because the constraint and the level curve exactly touch when you can't reduce (or increase) the function f(x,y) and still satisfy the constraint.

Let's do all this again with a different example:

Want to maximize f(x,y) = - [x^2 +y^2]

Let's plot the level curves:

Seems clear that the maximum occurs at (0,0). And the maximum f(0,0) is 0



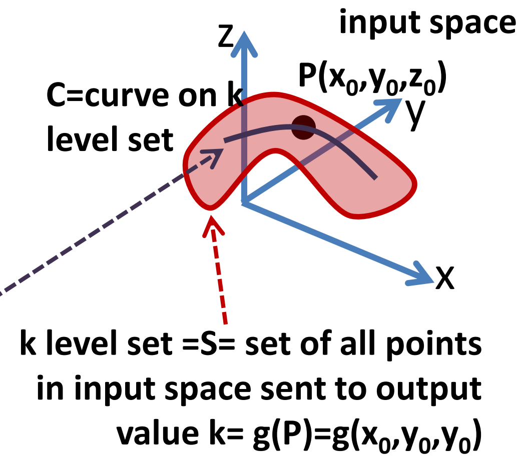
Let's now add a constraint. We require that (x,y) satisfy $y-x^2=2$. So if $g(x,y)=y-x^2$, we are requiring that g(x,y)=2

By adding this constraint, the maximum now occurs at (0,2), and the maximum value f(x,y)=-4

This maximum occurs where the constraint curve touches a level set exactly once!

Extreme value occurs where normal to the constraint curve points in the direction as the normal to the level set

Proof that: Extreme value occurs where normal to the constraint curve points in the same direction as the normal to the level set



Step 1: The constraint function g(x,y,z) has a level surface S going through the point P with level set value $k = g(x_0,y_0,z_0)$

Step 2: Suppose f(x,y,z) has an extreme point at P. This means that f has a max or min as you move along the level set.

Step 3: Let C: r(t)=(x(t),y(t),z(t)) be a curve on the k level set that passes through the point $P(x_0,y_0,z_0)$, with $r(t=t_0) = P$

Step 4: Let h(t) evaluates the objective function f along the curve C: h(t) = f(x(t),y(t),z(t))
So h(t) parameterizes by t the output of the objective function as you move along C

Step 5: So h(t) has a extreme value (max or min) at $t=t_0$ since f has an extreme point at P.

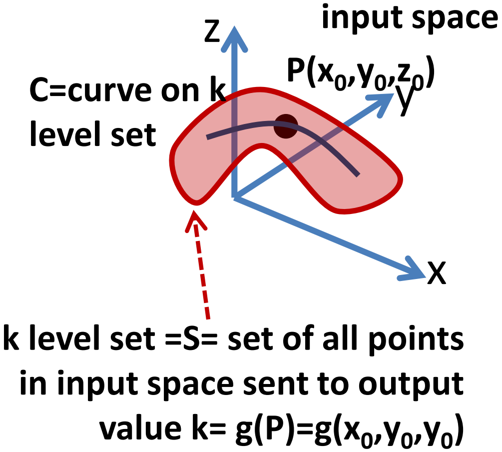
True because the point P is a critical point of f(x,y,z).

Step 6: So then $0 = \left.\frac{dh(t)}{dt}\right|_{t=t_0} = f_x(x_0,y_0,z_0)\frac{dx}{dt} + f_y(x_0,y_0,z_0)\frac{dy}{dt} + f_z(x_0,y_0,z_0)\frac{dz}{dt} = \overrightarrow{\nabla f}(x_0,y_0,z_0) \cdot \vec{r}'(t_0)$

Step 7: So $\overrightarrow{\nabla f}(x_0,y_0,z_0)$ **is orthogonal to any curve on S.**

Step 8: And we also know that $\overrightarrow{\nabla g}(x_0,y_0,z_0)$ **is orthogonal to the level set surface S.**

Step 9: So $\overrightarrow{\nabla f}(x_0,y_0,z_0)$ **and** $\overrightarrow{\nabla g}(x_0,y_0,z_0)$ **must point in the same direction!**



To summarize:

Suppose we want to find the extreme points of a function f(x,y,z) subject to the constraint g(x,y,z)=k

Then at an extreme point we must have that they point in the same direction, so we must have that

$$\overrightarrow{\nabla} f(x_0, y_0, z_0) = \lambda \overrightarrow{\nabla} g(x_0, y_0, z_0)$$

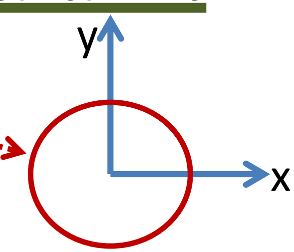
(they can have different lengths, but must point in the same direction)

The value λ is known as the "Lagrange multiplier"

Suppose we want to find the extreme points of a function f(x,y,z) subject to the constraint g(x,y,z)=k. Then at an extreme point we must have that they point in the same direction, so we must have that $\vec{\nabla} f(x_0, y_0, z_0) = \lambda \vec{\nabla} g(x_0, y_0, z_0)$

Example: Find the extreme values of f(x,y) = $x^2 + 2y^2$ subject to the constraint g(x,y)=x^2+y^2=1

In other words, walking along the red circle given by $x^2+y^2=1$ what are the maxima and minima of f(x,y)=x^2+2y^2 ?



f(x,y)=x^2+2y^2

Step 1: So we want to find the points (x_0,y_0) where $\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0)$

Step 2: $\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0)$ **is a vector equation, so we need to solve**

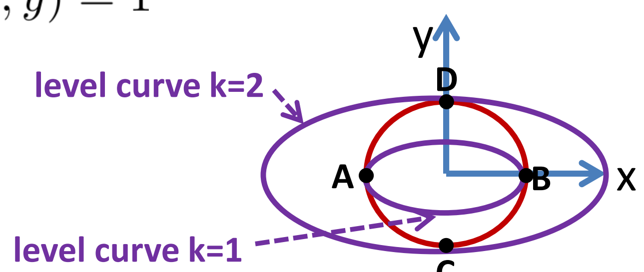
$$(f_x, f_y) = \lambda(g_x, g_y) \quad \text{plus the constraint} \quad g(x,y) = 1$$

$$f_x = \lambda g_x \rightarrow 2x = \lambda 2x$$

$$f_y = \lambda g_y \rightarrow 4y = \lambda 2y$$

$$g(x,y) = 1 \rightarrow x^2 + y^2 = 1$$

Three equations three unknowns (x,y,λ)



Level curves of x^2+2y^2

Step 3: Solve: from $2x=\lambda 2x$, either x= 0 or λ=1

Case 1: x=0 ⟶ from constraint this means y=-1 or y=1: gives extreme points (0,-1) and (0,1)

Case 2: λ=1 ⟶ from 2nd eq. this means y=0, from constraint means x=-1 or 1: Ext: (-1,0), (1,0)

Four extreme points: A=(0,-1), B=(0,1) C=(-1,0), D=(1,0)

Suppose we want to find the extreme points of a function f(x,y,z) subject to the constraint g(x,y,z)=k. Then at an extreme point we must have that they point in the same direction, so we must have that $\overrightarrow{\nabla f}(x_0, y_0, z_0) = \lambda \overrightarrow{\nabla g}(x_0, y_0, z_0)$

Example: Find the extreme values of f(x,y) = $-x^2 - y^2$ subject to the constraint g(x,y)=$y-x^2$=2

Step 1: So we want to find the points (x_0,y_0) where $\overrightarrow{\nabla f}(x_0, y_0) = \lambda \overrightarrow{\nabla g}(x_0, y_0)$

Step 2: $\overrightarrow{\nabla f}(x_0, y_0) = \lambda \overrightarrow{\nabla g}(x_0, y_0)$ **is a vector equation, so we need to solve**

$$(f_x, f_y) = \lambda (g_x, g_y) \quad \text{plus the constraint} \quad g(x,y) = 2$$

Three equations three unknowns (x,y,λ)

$$f_x = \lambda g_x \to -2x = \lambda(-2x) \to \lambda = 1 \quad \text{or} \quad x = 0$$
$$f_y = \lambda g_y \to -2y = \lambda$$

Step 3: Let's look at the cases

Case 1: λ = 1 ⟶ y=-1/2 ⟶ and we're dead, since the constraint g(x,y)=$y-x^2$=2 can't be solved

Case 2: so all that's left is x=0, so y=2, which is what we got earlier!