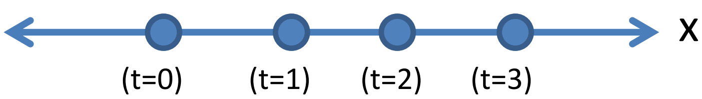
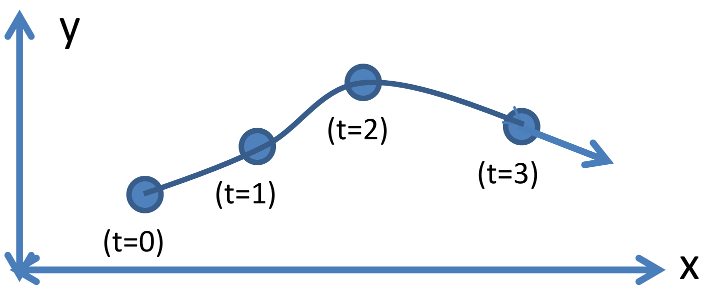
Review of First Third of the Course

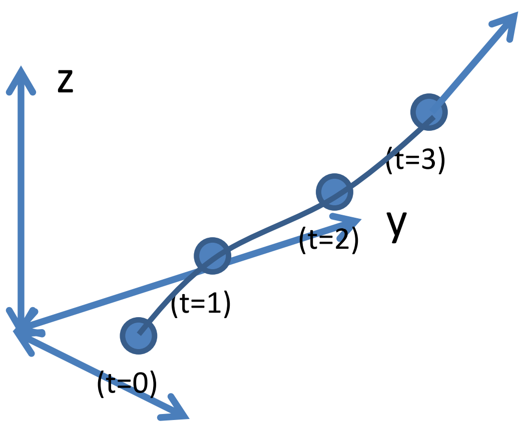
Parameterized curves



x = f(t) 1D parameterized function
Mapping from 1D to 1D
Gives x coordinate of object moving in 1D at time t.



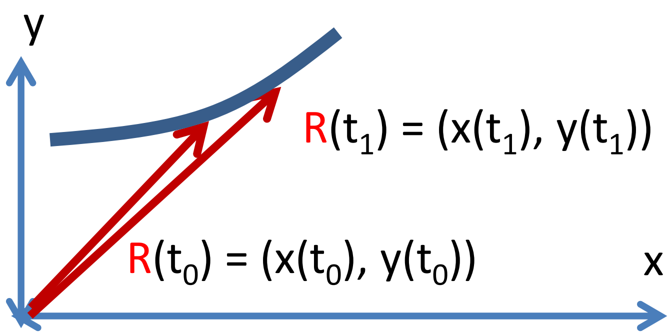
x = f(t), y=g(t) 1D parameterized function
Mapping from 1D to 2D
Gives (x,y) coordinate of object moving in 2D at time t.



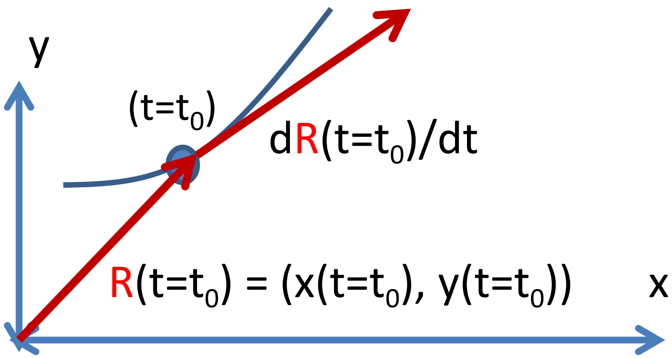
x = f(t), y=g(t), z=h(t) 1D parameterized function
Mapping from 1D to 3D
Gives (x,y,z) coordinate of object moving in 3D at time t.

Review of First Third of the Course

Derivatives of parameterized curves



$R(t) = (x(t), y(t))$ is vector description of curve in 2D.

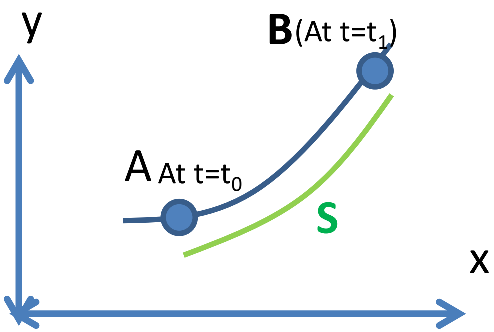


$dR(t)/dt=(dx(t)/dt\ ,\ dy(t)/dt)$ is vector derivative of particle trajectory

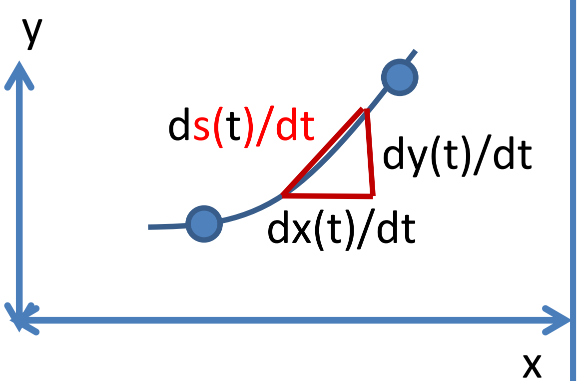
It is tangent to the curve at the point $x(t=t_0), y(t=t_0)$

Review of First Third of the Course

Arc-length of parameterized curves



(x(t), y(t)) describes curve in 2D.
What is the total arc-length **S** from A to B?



$dS(t)/dt = [\ (dx(t)/dt)^2 + (dy(t)/dt)^2\ ]^{(1/2)}$ using “bob’s” theorem.

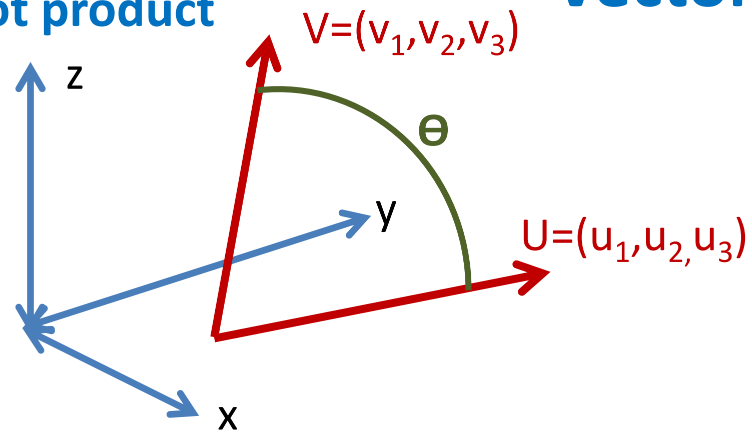
Add them all up to get total arc-length

$$S = \int_{t_0}^{t_1} [ds(t)/dt]\ dt$$

$$S = \int_{t_0}^{t_1} = \left[\ (dx(t)/dt)^2 + (dy(t)/dt)^2\ \right]^{(1/2)} dt$$

Vectors

Dot product



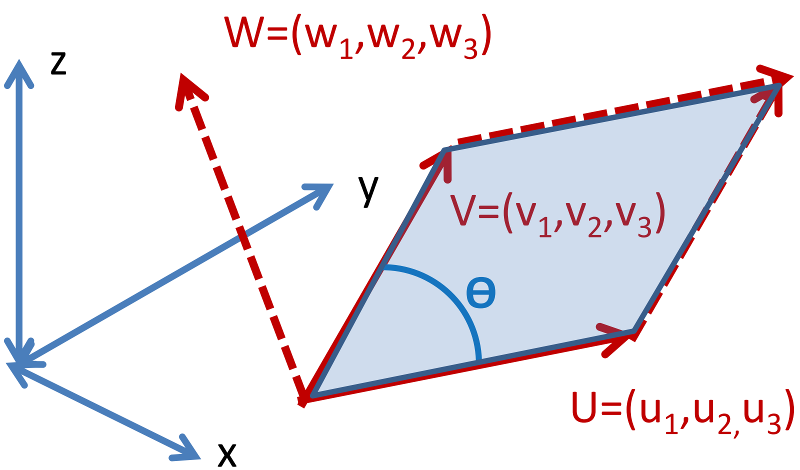
Two vectors in 3D: angle between is θ

Definition of dot product:
U dot **V** = $(u_1*v_1 + u_2*v_2 + u_3*v_3)$

Proved: U dot **V = |U| |V| cos θ**

Proved:
Two vectors are orthogonal ⟺ **U** dot **V=0**

Cross product



Given two vectors in 3 dimensions

Can define their cross product as **U** x **V** =
W=$(u_2*v_3-u_3*v_2, u_3*v_1-u_1*v_3, u_1*v_2-u_2*v_1)$

Proved: W=U x V is orthogonal to U and to V

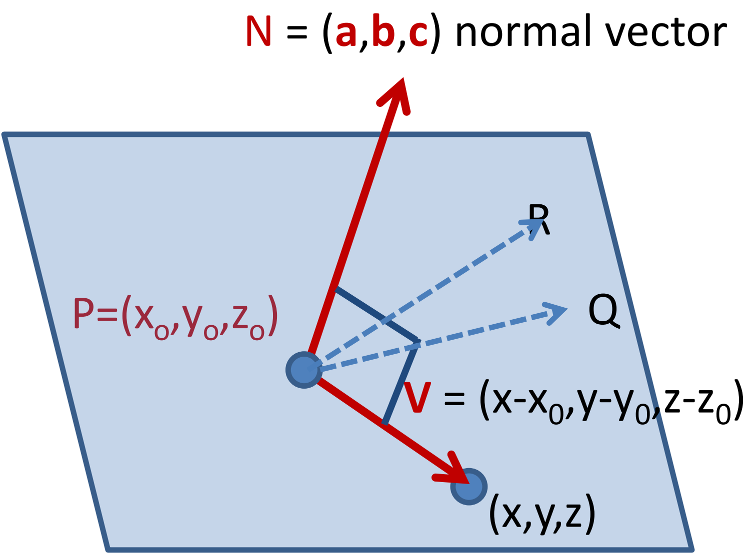
Proved: |U x V| = |U||V| sin θ

Proved: |U x V| = area of parallelogram

Defining planes using vectors

A plane is completely described if you are given a point $P=(x_0,y_0,z_0)$ and normal vector **N**

For any point (x,y,z) in the plane, the vector that connects (x,y,z) to P is given by **V** = $(x-x_0,y-y_0,z-z_0)$



We can write an equation for any point (x,y,z) in the plane by realizing that any such vector **V** in the plane must be orthogonal to **N**: that is, **N** dot **V** = 0

So any point (x,y,z) in the plane must satisfy **N** dot **V** = 0: which is the same as (a,b,c) dot $(x-x_0,y-y_0,z-z_0) = 0$ which becomes

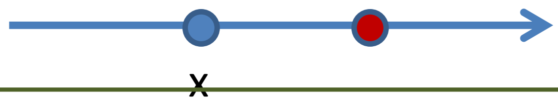
$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

If we are not given a normal vector N , but instead have three points P, Q, and R in the plane, we can find a normal vector N by taking the cross product

$$N = (Q-P) \times (R-P)$$

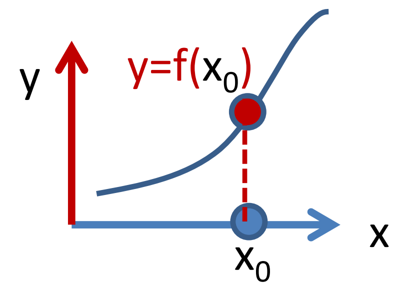
Functions of more than one variable

Input space | Need an extra dimension to graph it



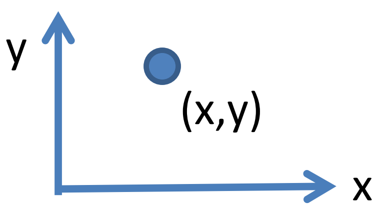
f(x): 1D input to 1D output

Function of one variable



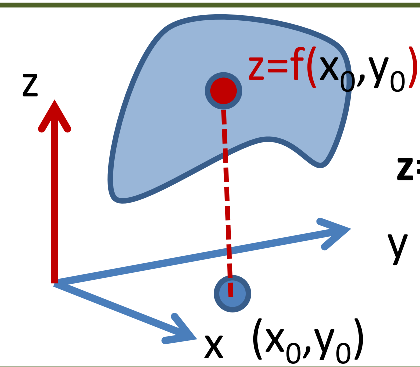
y=f(x)

(Red arrow is output space)

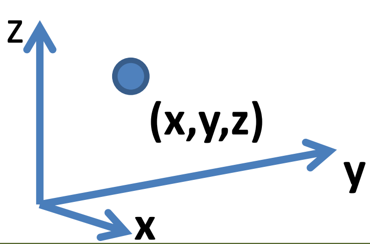


f(x,y): 2D input to 1D output

Function of two variables

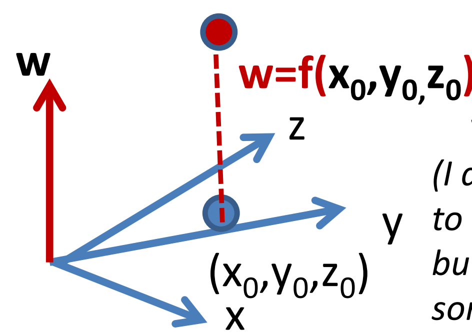


z=f(x,y)



f(x,y,z): 3D input to 1D output

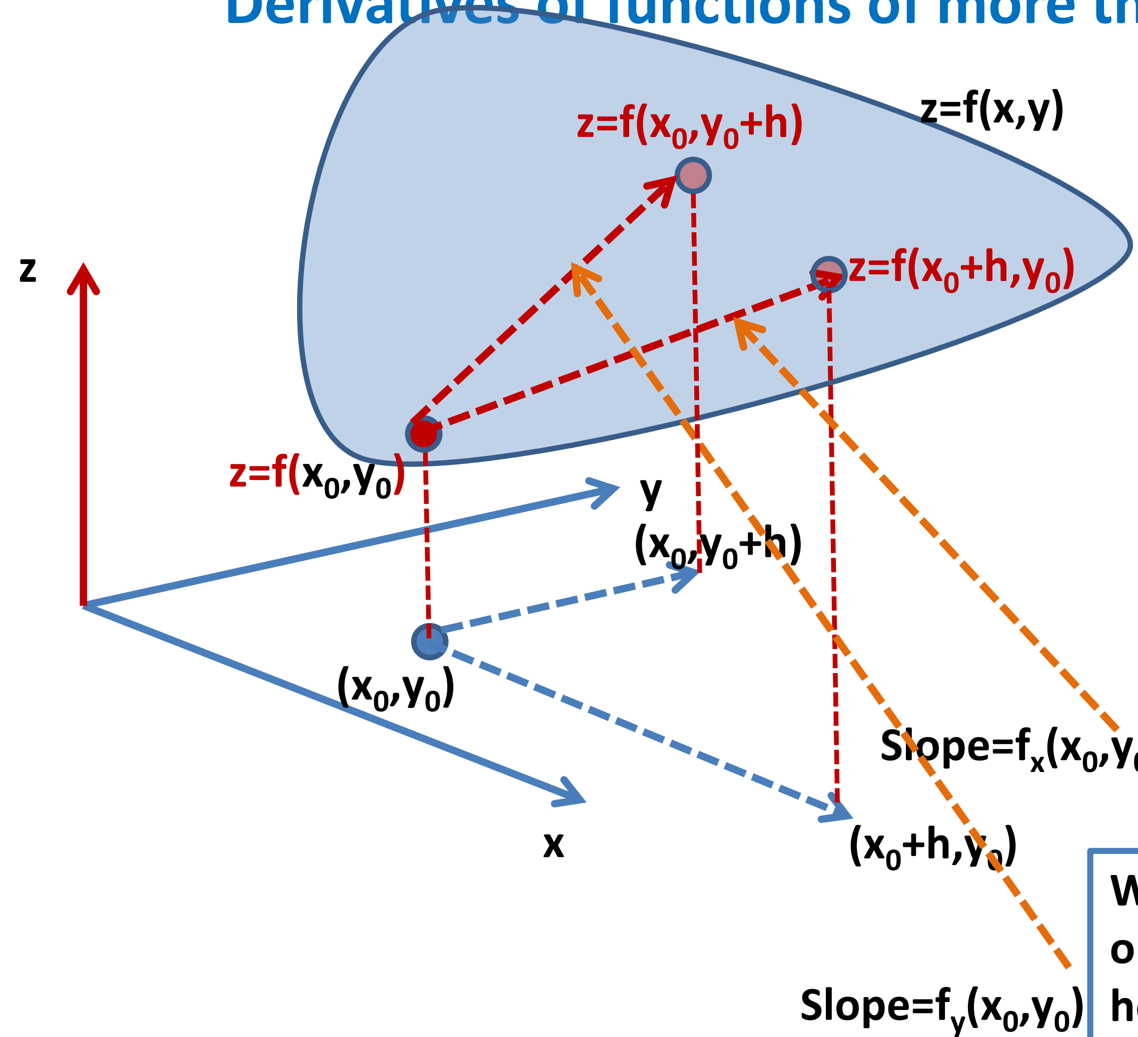
Function of three variables



w=f(x,y,z)

(I don't know how to draw the graph, but it's there somewhere!)

Derivatives of functions of more than one variable



Start with a function f(x,y)
The graph is 2D surface in R^3

At an input point (x_0,y_0) the function has an output $f(x_0,y_0)$

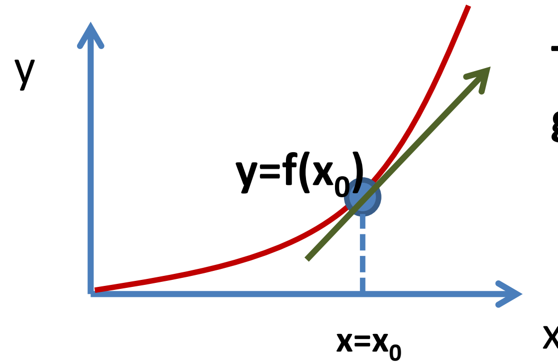
We can ask "how does the output change as x increases, holding y fixed?"

The slope of the line connecting these changes is $f_x(x_0,y_0)$

We can also ask "how does the output change as y increases, holding x fixed?" Answer= $f_y(x_0,y_0)$

Section 14.4: Tangent Planes: WHEN YOU ARE GRAPHING OUTPUT AGAINST INPUT!!!!!!

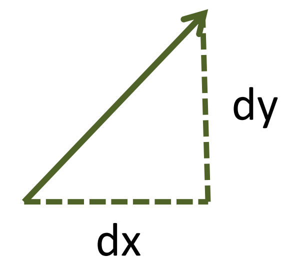
Let's recall 1D Calculus: y=f(x)



Tangent line with slope $f'(x_0)$ going through the point $(x_0,f(x_0))$

Tangent line at $(x_0,f(x_0))$ touches the graph y=f(x) at only one point near $(x_0,f(x_0))$

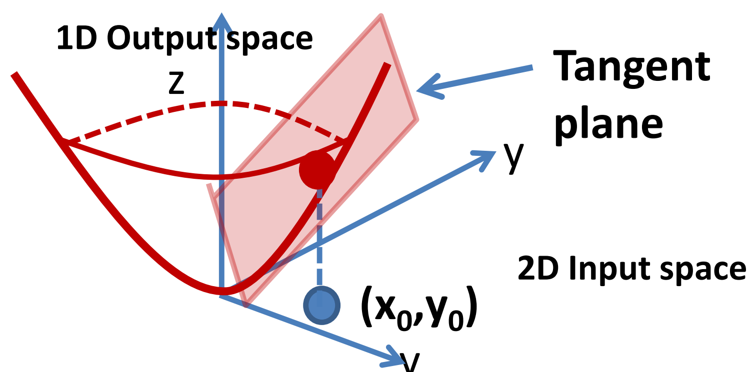
The tangent line is given by $y - f(x_0) = slope(x - x_0) = \left.\frac{df}{dx}\right|_{x_0} (x - x_0)$



Tangent vector = (dx,dy) = (1, dy/dx)= (1, f'(a))

We want to construct a similar idea for functions of two (or more variables):

The Tangent Plane



Tangent plane at $(x_0,y_0,f(x_0,y_0))$ touches the graph z=f(x,y) at only one point near $(x_0,y_0,f(x_0,y_0))$

Section 14.4: Tangent Planes: WHEN YOU ARE GRAPHING OUTPUT AGAINST INPUT!!!!!

1d: Slope of a tangent line

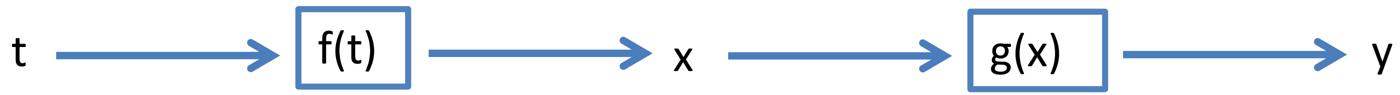
$$f(x) = f(a) + \left.\frac{df}{dx}\right|_a (x-a) \leftarrow \text{Equation for Line tangent to } a, f(a)$$

2d: Formula for the tangent plane

$$f(x,y) = f(a,b) + \left.\frac{\partial f}{\partial x}\right|_{a,b} (x-a) + \left.\frac{\partial f}{\partial y}\right|_{a,b} (y-b) \leftarrow \text{Equation for plane tangent to } a, b, (a,b)$$

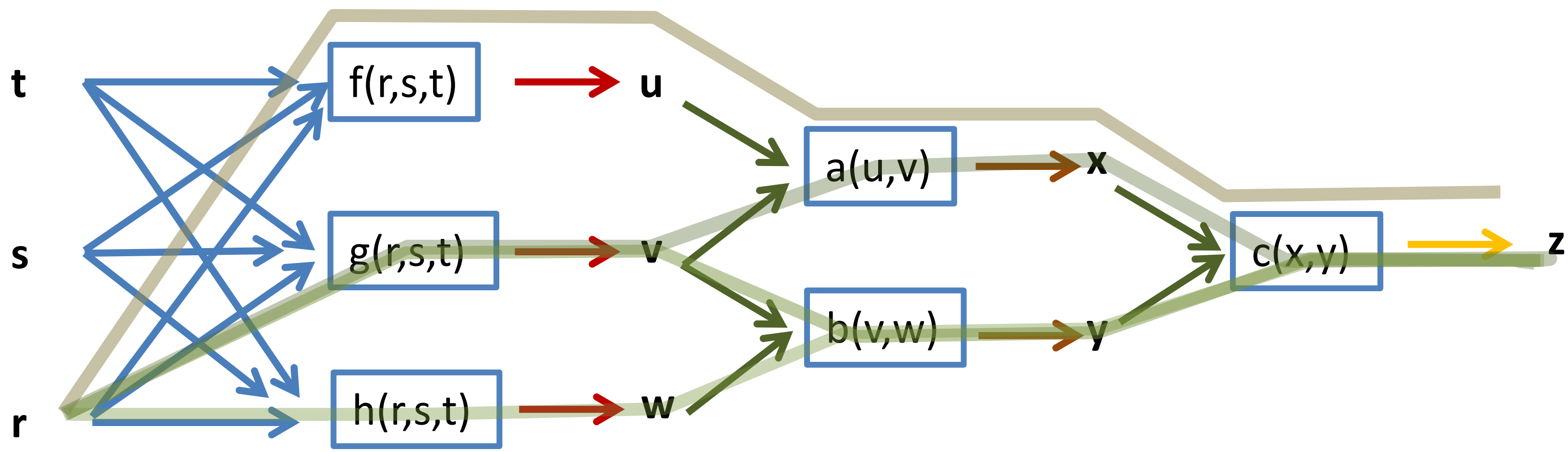
The Multi-Dimensional Chain Rule

Of one variable:



$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

Of many variables:



To find $\frac{\partial z}{\partial r}$ follow all pathways through the graph:

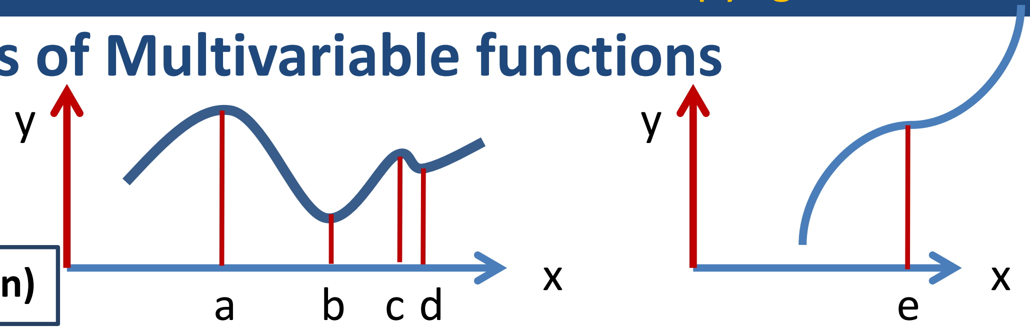
$$\frac{\partial z}{\partial r} =$$

Critical Values of Multivariable functions

One dimension y=f(x)

Critical points: Places where df/dx =0

If, in addition, $d^2f/dx^2 < 0$ (>0) then max (min)



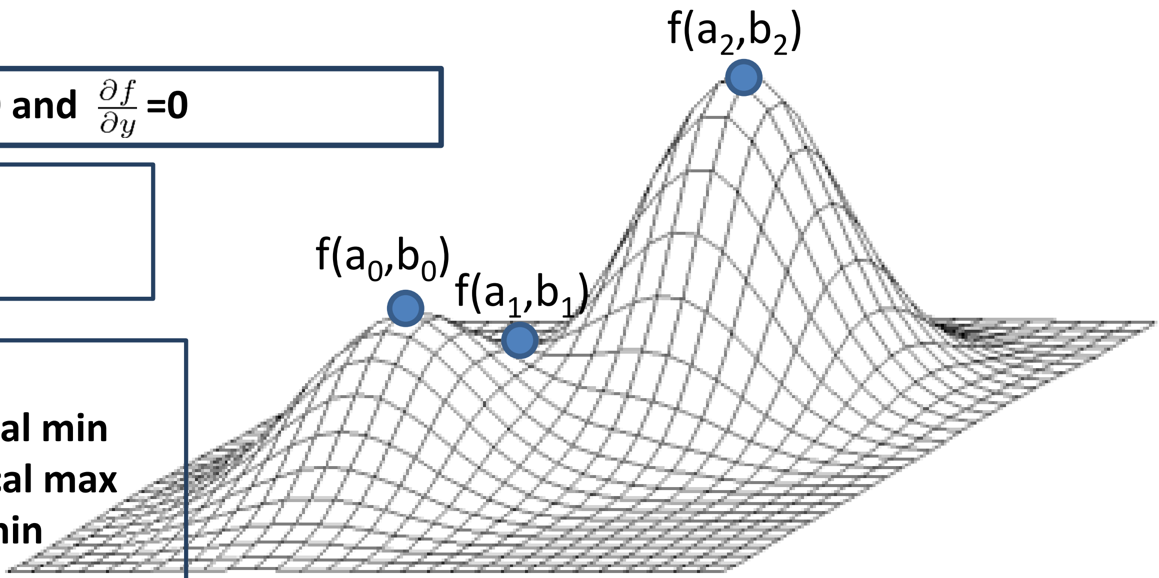
x=a: global max ||| x=b: global min ||| x=a,c: local maximum |||x=b,d: local min |||x=e: (inflection)

Two dimensions z=f(x,y)

Critical points: Places where both $\frac{\partial f}{\partial x}$ =0 and $\frac{\partial f}{\partial y}$ =0

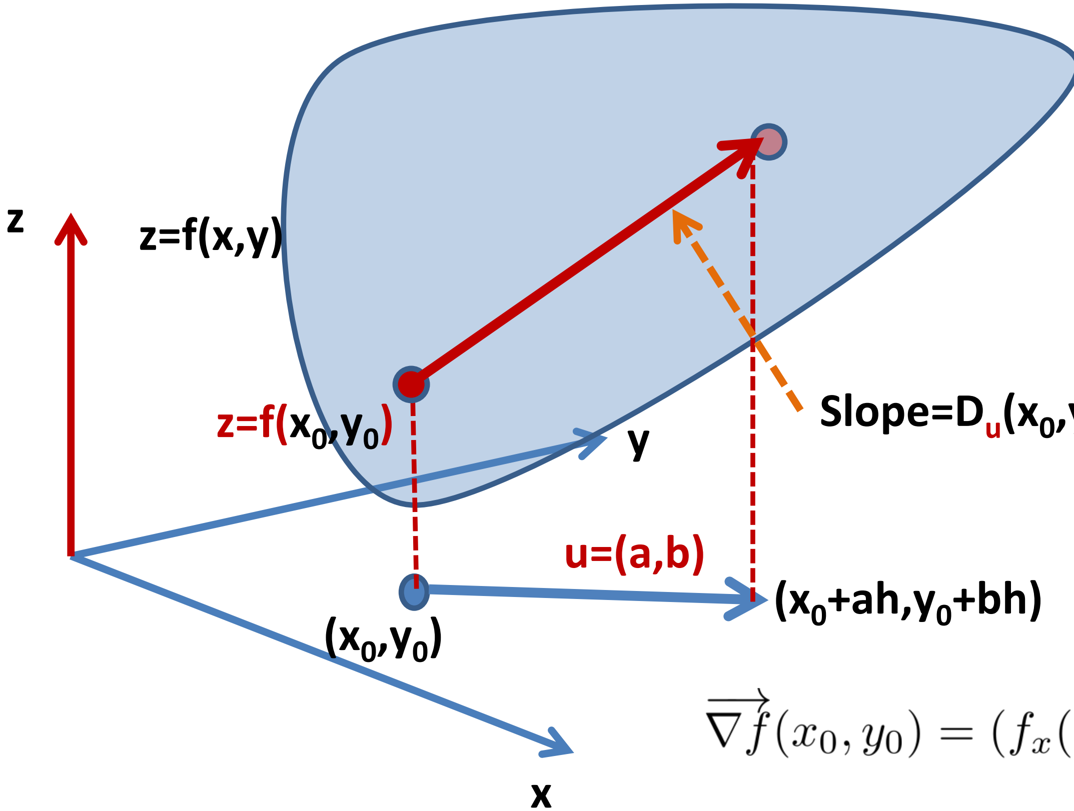
At the input point (a,b) define D as
$D(a,b)=f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$

Suppose (a,b) is a critical point, then
- **(a) if D(a,b)>0 and fxx(a,b)>0, then local min**
- **(b) if D(a,b)>0 and fxx(a,b)<0, then local max**
- **(c) if D(a,b)<0, then neither max nor min**



(a_0,b_0) = local max. (a_1,b_1) = neither (saddle) (a_2,b_2)= global max

Direction Derivatives



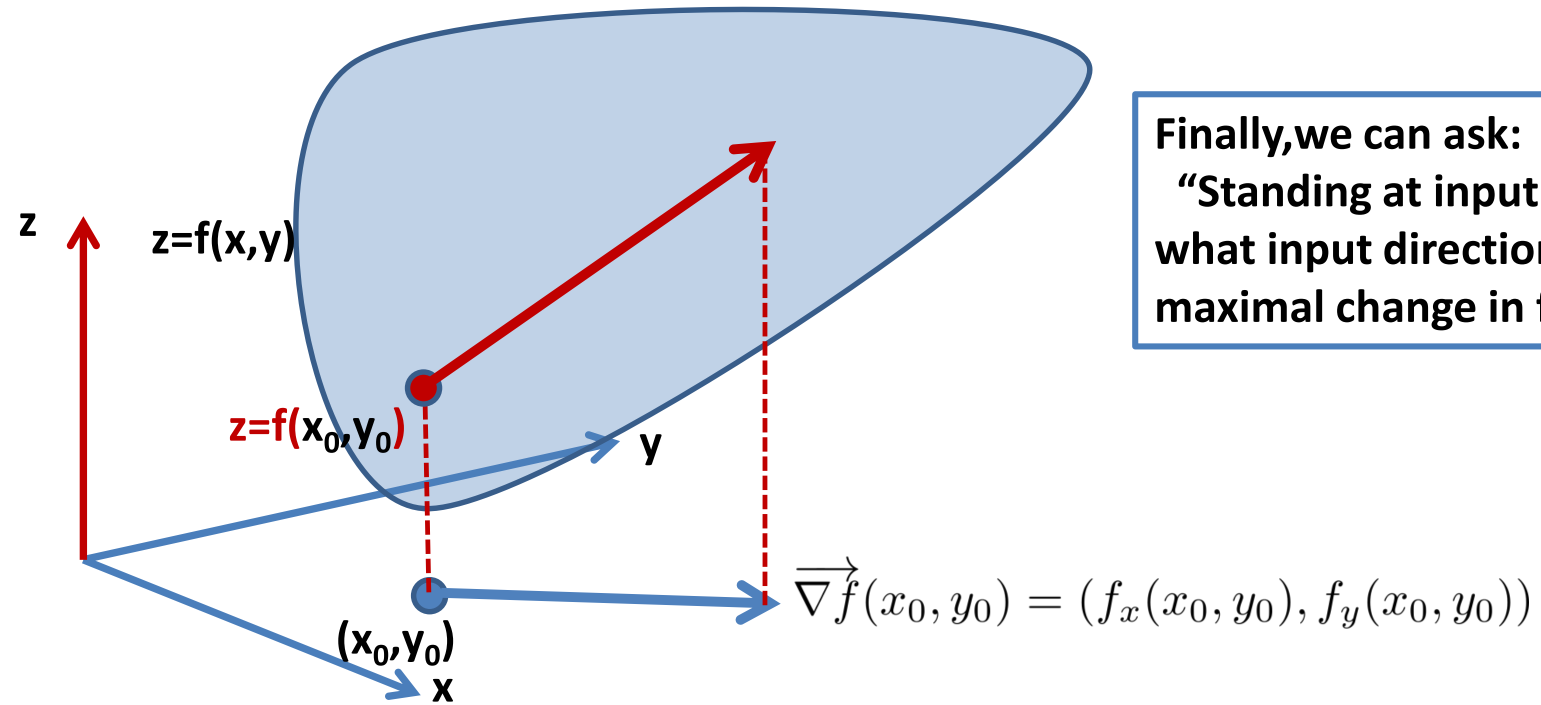
We can also ask: Standing at the input point (x_0,y_0), how does the output change if we move in the direction u=(a,b), where u is a unit vector?

$$\overrightarrow{\nabla} \vec{f}(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$$

And we proved that:

$$D_u(x_0,y_0)=a * f_x(x_0,y_0) + b * f_y(x_0,y_0) = \overrightarrow{\nabla} \vec{f} \cdot \vec{u} \quad \text{At } x_0,y_0$$

The gradient



Finally,we can ask:
"Standing at input point (x_0,y_0), what input direction gives the maximal change in f(x,y)?"

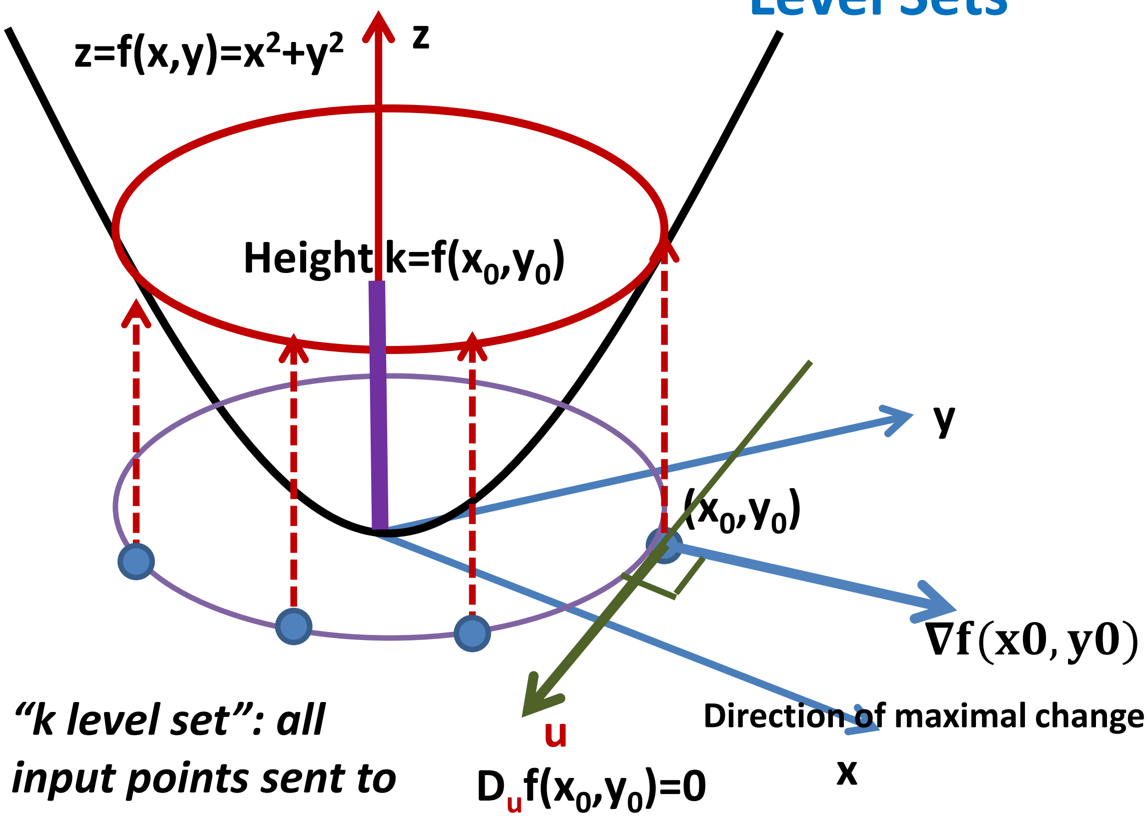
The answer is to move in the gradient direction

$$\overrightarrow{\nabla f}(x_0,y_0) = (f_x(x_0,y_0), f_y(x_0,y_0))$$

Why? Because $D_{\vec{u}} f(x_0,y_0) = \overrightarrow{\nabla f}(x_0,y_0) \cdot \vec{u} = |\overrightarrow{\nabla f}||\vec{u}|\cos\theta$ **is biggest when** $\cos\theta = 1$

And when $\cos\theta = 1$, then $\theta = 0$ so ∇f and $\vec{u}$ point in the same direction

Level Sets



"k level set": all input points sent to the same output k

Start with a function f(x,y)
The graph is 2D surface in R^3

Level set with value k is the set of all input points sent to the value k.

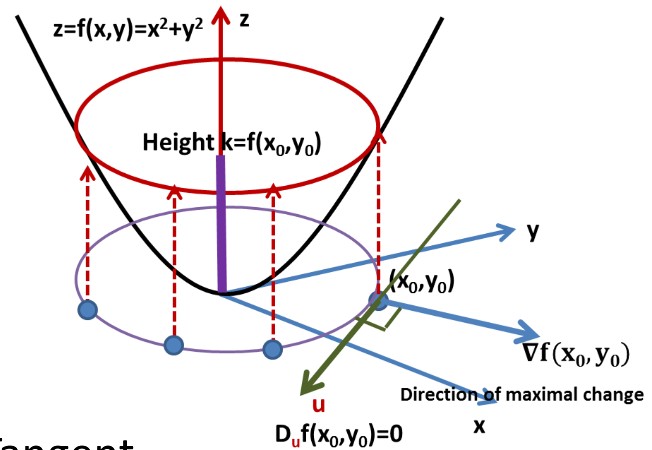
Moving along the k level set in input space, the function f(x,y) doesn't change

So, if u is a direction along the k level set, then $D_u f(x_0,y_0)=0$

The gradient at $f(x_0,y_0)$ is the direction of maximal change

Since $D_u f(x_0,y_0)=0$ and since $0=D_u f(x_0,y_0)= \nabla f(x0, y0)$ dot **u**
Then $u(x_0,y_0)$ is perpendicular to $\nabla f(x0, y0)$
So the gradient $\nabla f(x0, y0)$ is normal to the tangent to the level set going through (x_0, y_0)

Level Sets in Higher Dimensions

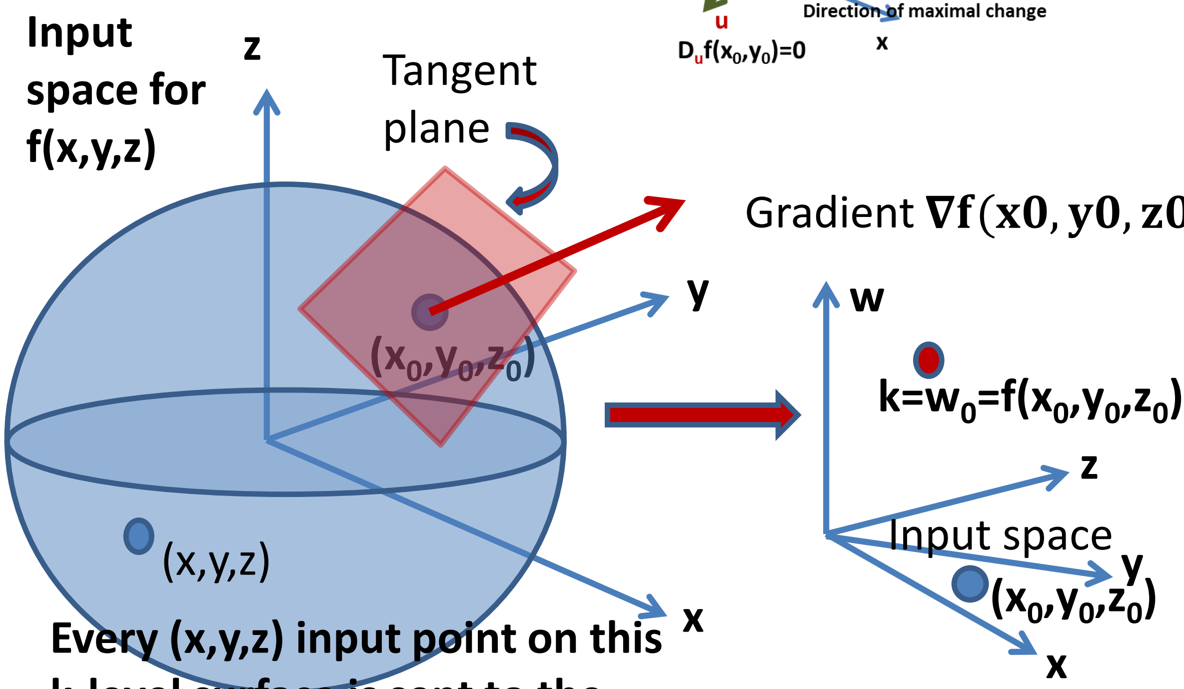


Again. At input (x_0,y_0) the gradient is normal to the line tangent to the $k=f(x_0,y_0)$ level curve passing through (x_0,y_0).

This same idea is true in higher dimensions. For a function $w=f(x,y,z)$, the input space is three-dimensional.

At input (x_0,y_0,z_0), the gradient is normal to the *plane tangent* to the $k=f(x_0,y_0,z_0)$ level set passing through (x_0,y_0,z_0).

So we have an equation for the tangent plane at the point (x_0,y_0,z_0), since we know a point and the normal $\nabla f(x0,y0,z0)$



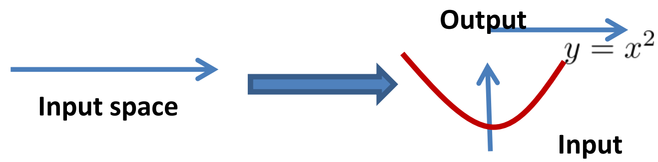
Every (x,y,z) input point on this k-level surface is sent to the same output $k=f(x_0,y_0,z_0)$

Graph of output vs. input

Finally, there is some (understandable) confusion about tangent planes. Let me try to sort it out

Suppose y=f(x). Find the tangent at input x_0 **Example: $y=x^2$ $x_0 = 3$** ***Two* ways to do this:**

Way #1: Interpret y=f(x) as the graph of output against input:



A mapping from $\mathbb{R}^1 \rightarrow \mathbb{R}^1$

Step 1: Draw the output against input

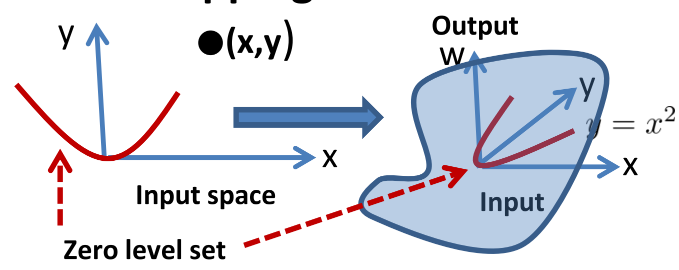
Step 2: Want the tangent point at (3,f(3)) = (3,9)

Step 3: We need the slope to use the formula $(y - y_0) = (slope) * (x - x_0)$

Step 4: Slope is = df/dx = 2x: at input x_0=3, slope is 6.

So answer is y-9 = 6(x-3) which we can rewrite as y = 6x - 9

Way #2: Consider a new function w(x,y) = y-f(x): w(x,y)=y-x^2 is a mapping from $\mathbb{R}^2 \rightarrow \mathbb{R}^1$



Step 1: Realize that the zero level set of function w(x,y) is the red curve, which is the set of all (x,y) sent to output 0, in other words, whenever g=f(x)

Step 2: So, the input x_0=3, y_0=9 gets sent to w = 0

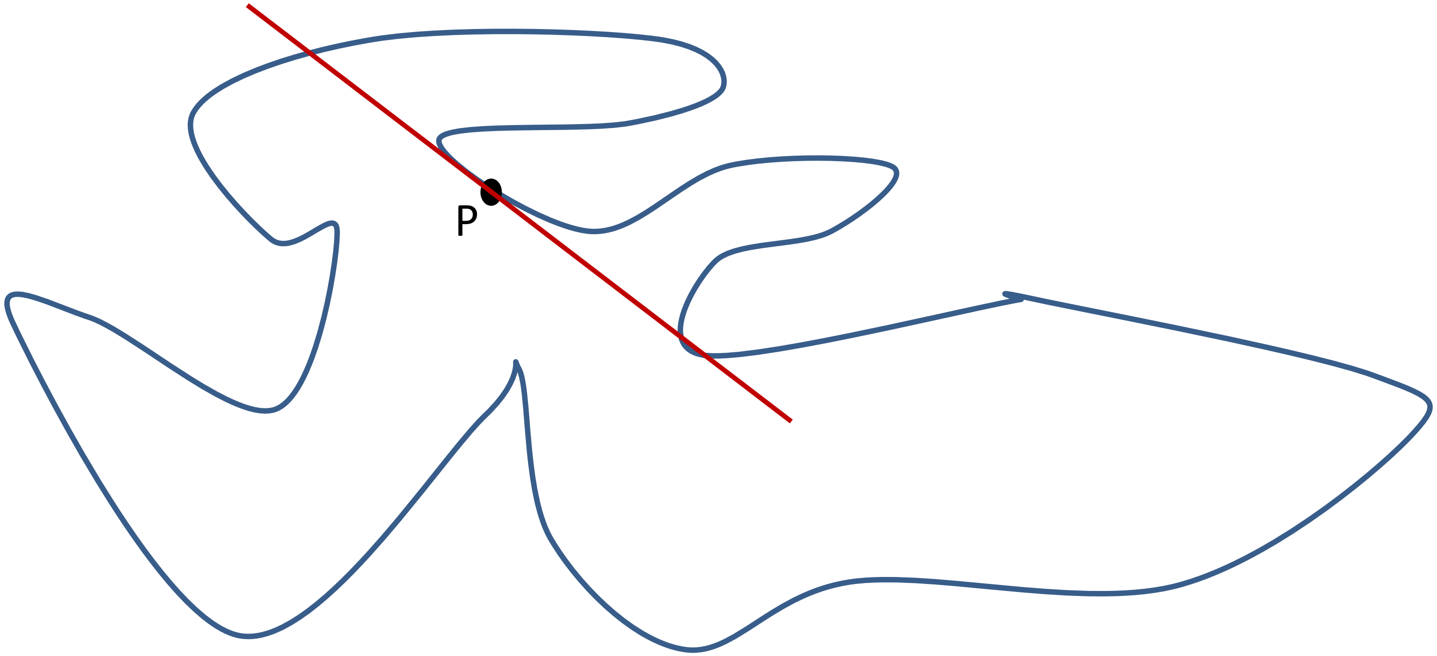
Step 3: That means that at the input (3,9) the gradient $\nabla w(x,y) = (-2x, 1) = (-6, 1)$ **is normal to the tangent to the level set 0=y-x^2**

Step 4: So, using the normal equation for a line $\vec{n} \cdot (x - x_0, y - y_0) = 0$

$$(-6,1) \cdot (x-3, y-9) = 0 \rightarrow -6x + 18 + y - 9 = 0 \rightarrow y = 6x - 9$$

Final question: Why would you ever use the second way?

Answer: because you might get asked to find the tangent to the following funky curve at point P –it's not a function….



Example: Find the tangent to w=f(x,y,z) = $x^2 + y^2 + z^2$ at the input (1,2,3)

Way #1: Interpret w=f(x,y,z) as the graph of output against input: A mapping from $\mathbb{R}^3 \to \mathbb{R}^1$

Step 1: Want the tangent object at the point (1,2,3,f(1,2,3)) = (1,2,3,14)

Step 2: We need the various partials to use the formula

$$f(x,y,z) = f(a,b,c) + \left.\frac{\partial f}{\partial x}\right|_{a,b,c}(x-a) + \left.\frac{\partial f}{\partial y}\right|_{a,b,c}(y-b) + \left.\frac{\partial f}{\partial z}\right|_{a,b,c}(z-c)$$

Equation for hyperplane tangent to $f(a,b,c)$

Step 4: f_x=2x, f_y=2y,f_z=2z---so (f_x,f_y,f_z) =(2,4,6).

So tangent object is w=14 + (2)(x-1)+(4)(y-2)+(6)(z-3)= 2x+4y+6z+14

Way #2: Think of the equation w=f(x,y,z) as a particular level set of a new function

U(x,y,z,w) = w-f(x,y,z)=w-(x^2+y^2+z^2) is a mapping from $\mathbb{R}^4 \to \mathbb{R}^1$

Step 1: Realize that the zero level set of function U(x,y,z,w) is the set of all (x,y,z,w) sent to output 0

Step 2: So, the input x_0=1, y_0=2, z_0=3, w_0=14 gets sent to U = 0

Step 3: That means that at the input (1,2,3,14) the gradient is normal to the tangent to the level set 0= w-(x^2+y^2+z^2)

Step 4: The gradient at input (1,2,3,14) is

$$\nabla U(x,y,z,w) = (-2x,-2y,-2z,1) = (-2,-4,-6,1) = \vec{n}$$

$$\vec{n}\cdot(x-x_0, y-y_0, z-z_0, w-w_0) = 0$$

$$(-2,-4,-6,1)\cdot(x-1, y-2, z-3, w-14) = 0 \to w = 2x+4y+6z+14$$