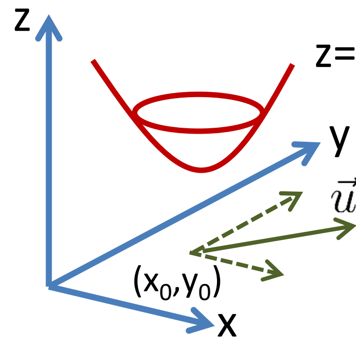
Review of last lecture:

(1) Definition: $D_{\vec{u}} f(x_0, y_0) \equiv \lim_{h \to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0, y_0)}{h}$

where u is a unit vector with components

$$\vec{u} = (a, b) \qquad |\vec{u}| = 1$$



(2) And we found that:

$$D_u f(x, y) = \overrightarrow{\nabla f} \cdot \vec{u}$$

(3) What direction makes $D_u f(x, y) = \overrightarrow{\nabla f} \cdot \vec{u}$ **the biggest?**

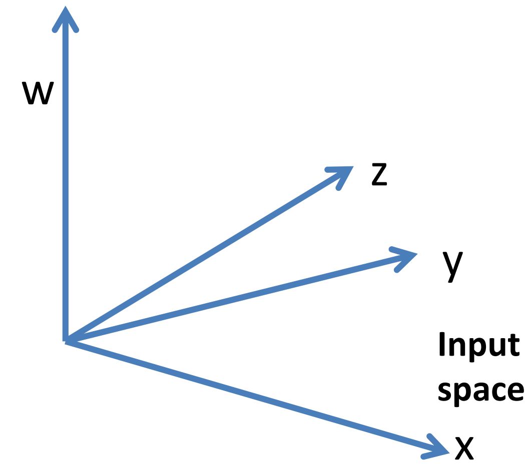
Since $D_u f(x, y) = \overrightarrow{\nabla f} \cdot \vec{u} = |\overrightarrow{\nabla f}| \; |\vec{u}| \cos\theta$

This is biggest when cos θ = 1 which happens when

$\overrightarrow{\nabla f}$ and $\vec{u}$ **point in the same direction!**

Let's use these new ideas to think about tangent planes again

$$D_u f(x,y) = \overrightarrow{\nabla f} \cdot \vec{u}$$



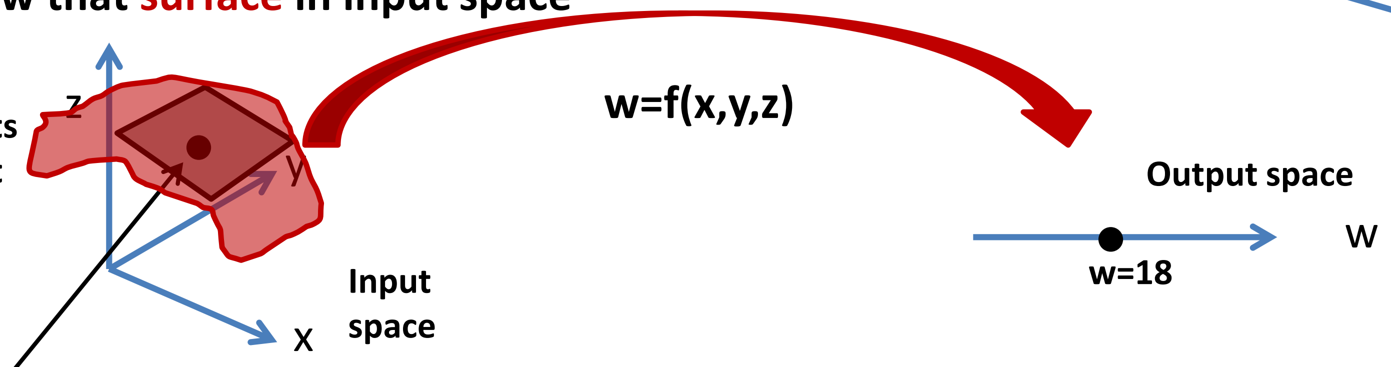
(1) **Consider a function of 3 variables: w = f(x,y,z)** $\mathbb{R}^3 \rightarrow \mathbb{R}^1$

(2) **Requires four dimensions to graph**

(3) **Let's look at the level set 18 = f(x,y,z) which is the set of all input triples (x,y,z) that get sent to the output w=18**

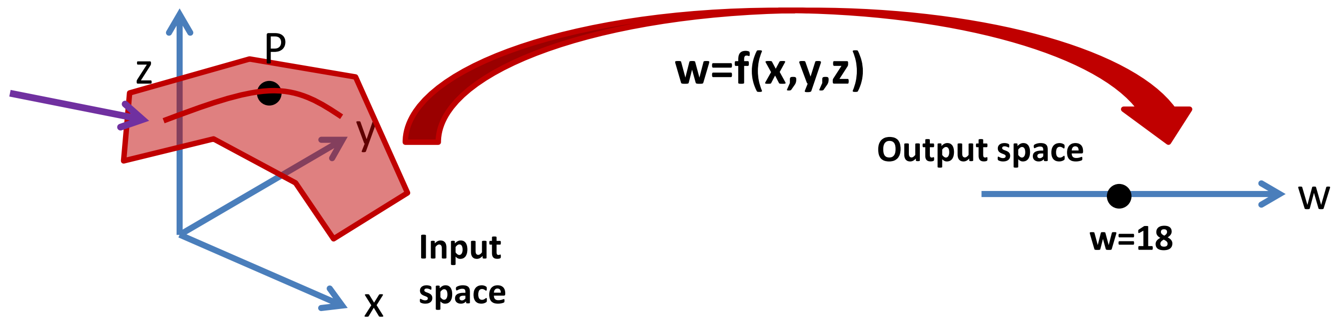
(4) **We can draw that surface in input space**

Surface of all points in input space sent to 18



(5) Let's find a nice expression for the tangent plane to this level surface at input point $P=(x_0,y_0,z_0)$

Step 1: Imagine a curve $\vec{r}(t)=(x(t),y(t),z(t))$ on level surface: all inputs sent to output 18



So f(x(t),y(t),z(t)) = 18

Step 2: Let's differentiate both sides with respect to t

$$\frac{d}{dt}\left[f(x(t),y(t),z(t))\right] = \frac{d}{dt}\left[18\right] \Longrightarrow \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$$

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = 0 \Longrightarrow \overrightarrow{\nabla f} \cdot \vec{r}\,'(t) = 0$$

At $t=t_0$ $\overrightarrow{\nabla f}(t_0) \cdot \vec{r}\,'(t_0) = 0$

Step 3: $\overrightarrow{\nabla f}(t_0) \cdot \vec{r}\,'(t_0) = 0$ **says that the gradient is perpendicular to the tangent to every curve in the 18 level set passing through P = (x_0,y_0,z_0)**

Step 4: Ah ha! So $\overrightarrow{\nabla f}(x_0,y_0,z_0)$ **is normal to the tangent plane at the point P=(x_0,y_0,z_0)**

Step 5: Remember that the tangent plane is

$$\boxed{a(x-x_0)+b(y-y_0)+c(z-z_0)=0}$$

So here is our snappy new formula for a plane tangent to the level surface at the point (x_0,y_0,z_0)

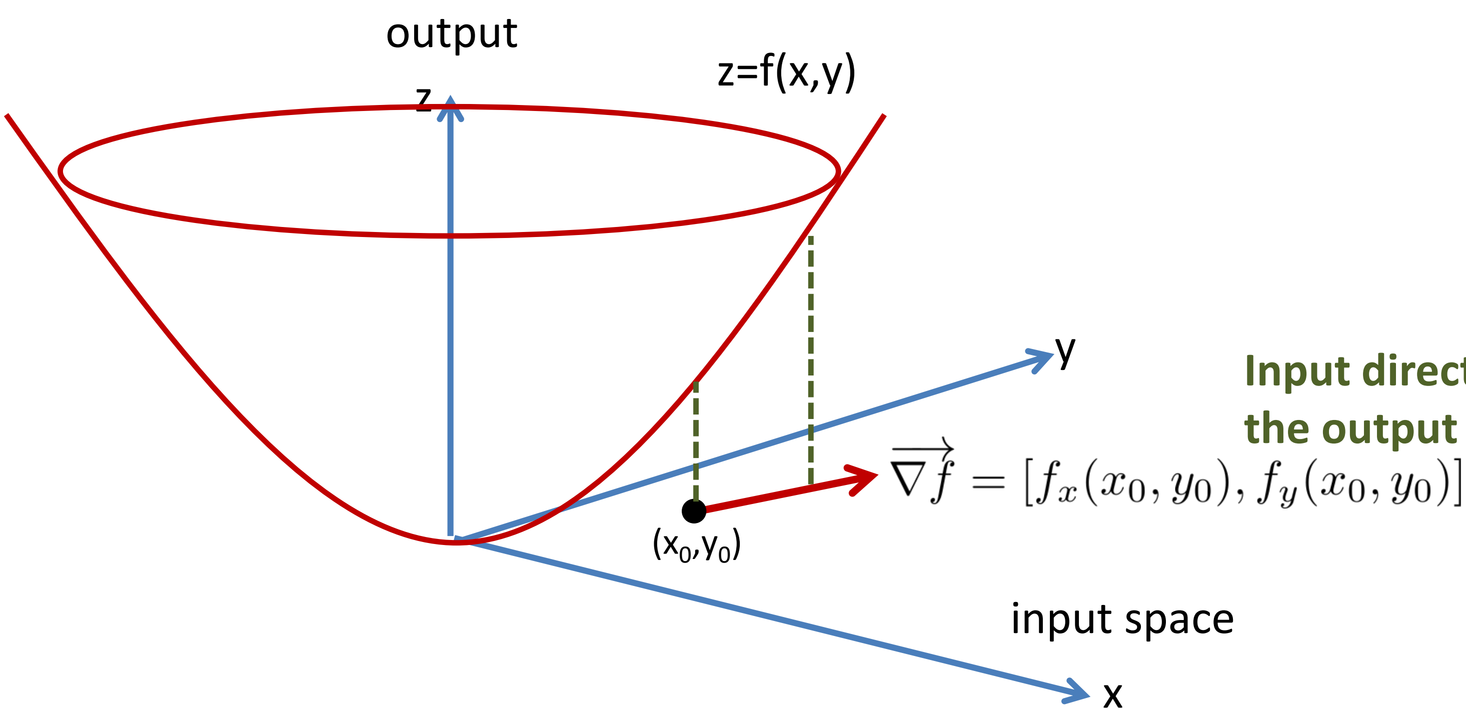
$$\overrightarrow{\nabla f}(x_0,y_0,z_0)\cdot(x-x_0,y-y_0,z-z_0)=0$$

$$f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+f_z(x_0,y_0,z_0)(z-z_0)=0$$

Section 14.6: Let's review what just happened: The gradient and tangent planes

So far, we have defined: $D_{\vec{u}} f(x,y) = \overrightarrow{\nabla f} \cdot \vec{u}$**= [standing at input point (x,y), this measures the change in f as we move in the direction** $\vec{u}$ **]**

And, we have found that f changes the most if we move in the same direction $\vec{u}$ **as the gradient** $\overrightarrow{\nabla f}$



Input direction that changes the output the most

Section 14.6: The gradient and tangent planes

> **Claim: for any function f(x,y) or f(x,y,z), [more generally, f: $\mathbb{R}^n \rightarrow \mathbb{R}^1$]**
> **The gradient at input (x_0,y_0,z_0) is orthogonal to the level set passing through (x_0,y_0,z_0)**

Wow! What does this even mean?

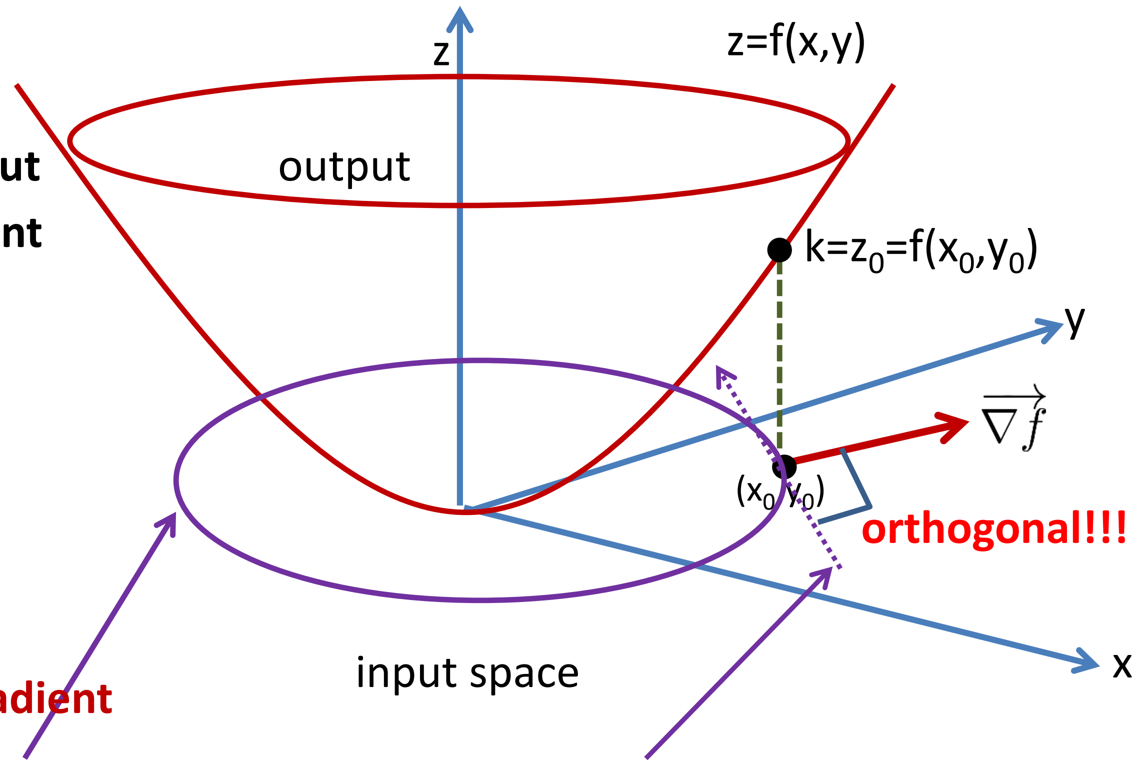
(1) We have a function

(2) At an input (x_0,y_0) we have an output

(3) At an input (x_0,y_0) we have a gradient

(4) At an input (x_0,y_0) we also have a level set corresponding to all inputs sent to the same output.

(5) And we have a tangent to that level set at the input (x_0,y_0)



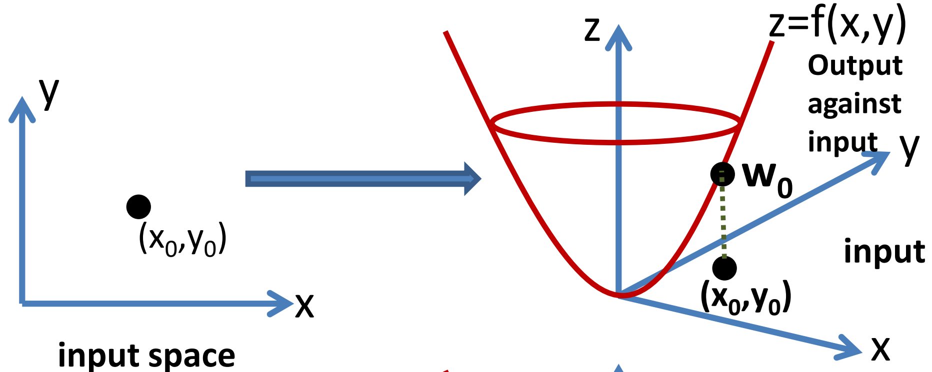
And I am claiming that tangent and gradient are orthogonal!

k level set = set of all input points sent to output z=k

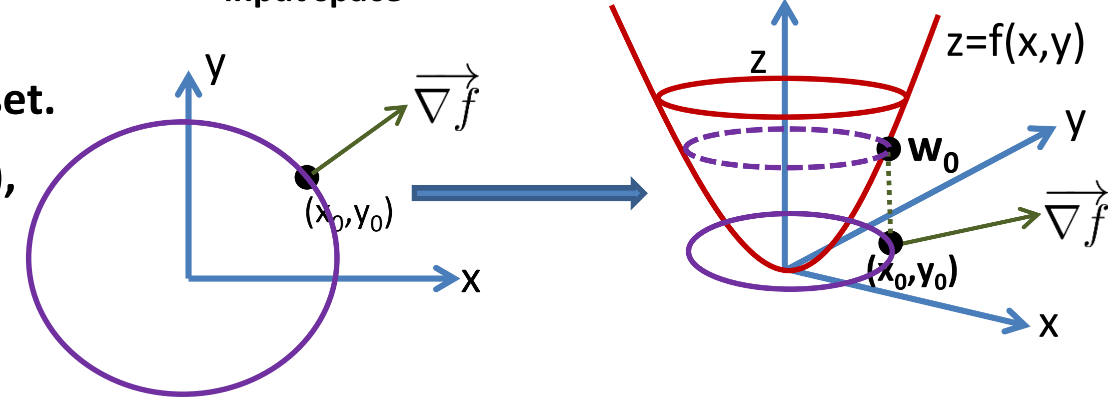
Tangent to level curve at (x_0,y_0)

Claim: for any function f(x,y) or f(x,y,z), [more generally, f: $\mathbb{R}^n \rightarrow \mathbb{R}^1$]
The gradient at input (x_0,y_0,z_0) is orthogonal to the level set passing through (x_0,y_0,z_0)

- **First, what does everything mean?**
- **Given a function f(x,y), input space is 2D**
- **Given a point (x_0,y_0) in input space, we evaluate the function at the input: that is, let $w_0 = f(x_0,y_0)$**



- **What other points besides (x_0,y_0) in input space get sent to output $w_0 = f(x_0,y_0)$? This would be the w_0 level set.**
- **We could also ask, standing at (x_0,y_0), [IN INPUT SPACE!!!] what direction increases the output the most? This is the gradient $\overrightarrow{\nabla f}$**



Here come the proof:

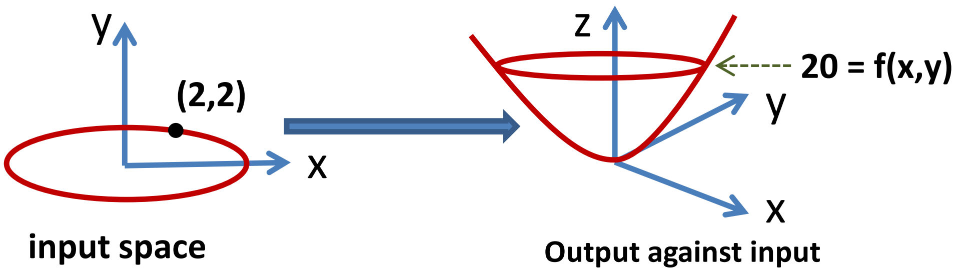
Step 1: Along the level curve, the output doesn't change, so $D_{\vec{u}} f(x_0,y_0) = 0$ **when** $\vec{u}$ **is tangent to the level curve at (x_0,y_0)**

Step 2: So $D_{\vec{u}} f(x_0,y_0) = \overrightarrow{\nabla f} \cdot \vec{u} = 0$ **so** $\vec{u}$ **must be orthogonal to** $\overrightarrow{\nabla f}$ **DONE!**

The gradient at input (x_0,y_0,z_0) is orthogonal to the level set passing through (x_0,y_0,z_0)

Example: Find the equation of the line tangent to the curve $x^2 + 4y^2 = 20$ at input point (2,2)

Solution: We attack this problem by turning the curve into the level set of a higher dimensional function. Once we do this, we will know that the tangent at (2,2) is orthogonal to the gradient.



Step 1: Let $f(x,y) = x^2 + 4y^2$

Step 2: Then the "20" level set is the set of all input points sent to output "20".

Step 3: So the gradient $\overrightarrow{\nabla f} = (2x, 8y) = (4, 16)$ **is orthogonal to the tangent line at input (2,2)**

Step 4: So the tangent line goes through (2,2), and is orthogonal to the normal vector (4,16)

Step 5: But wait! We know the formula for a line going through (x_0,y_0) with normal n=(a,b)

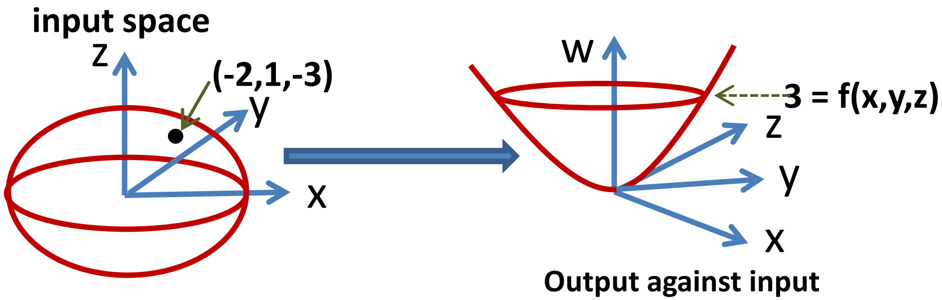
$$\vec{n} \cdot (x - x_0, y - y_0) = 0$$

$$4(x-2) + 16(y-2) = 0$$

The gradient at input (x_0,y_0,z_0) is orthogonal to the level set passing through (x_0,y_0,z_0) …and this all works in 3D as well (and higher)

Find the equation of plane tangent to the ellipsoid $x^2/4 + y^2 + z^2/9 = 3$ at input point (-2,1,-3)

Solution: We attack this problem by turning the ellipsoid into the level set of a higher dimensional function. Once we do this, we will know that the tangent plane at (-2,1,-3) is orthogonal to the gradient.



Step 1: Let $f(x,y,z) = x^2/4 + y^2 + z^2/9$

Step 2: Then the "3" level set is the set of all input points sent to output "3".

Step 3: So the gradient $\vec{\nabla f} = ((1/2)x, 2y, 2z/9)\Big|_{(-2,1,-3)} = (-1, 2, -2/3)$ **is orthogonal to the tangent plane at input (-2,1,-3)**

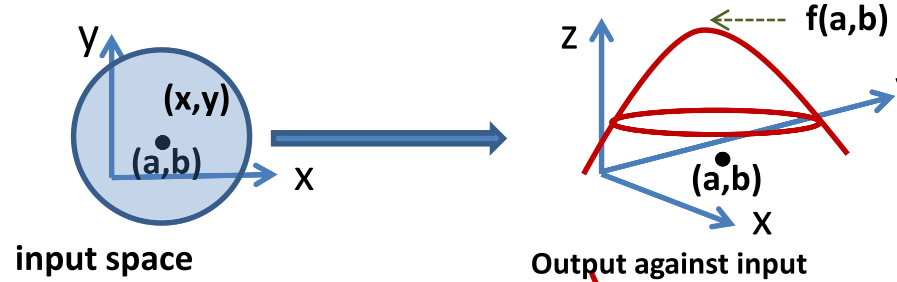
Step 4: So the tangent plane goes through (-2,1,-3), and is orthogonal to the normal vector (-1,2,-2/3)

Step 5: We know the formula for a plane going through (x_0,y_0,z_0) with normal n=(a,b,c)

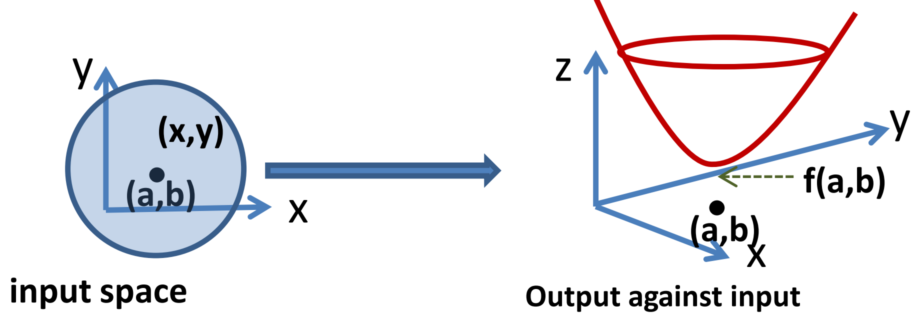
$$\vec{n} \cdot (x - x_0, y - y_0, z - z_0) = 0 \Longrightarrow -1(x+2) + 2(y-1) - 2/3(z+3) = 0$$

Section 14.7: Maximum and Minimum values of Functions

Definition: A function of two variables has a local maximum at (a,b) if f(a,b) ≥ f(x,y) when (x,y) is near (a,b)

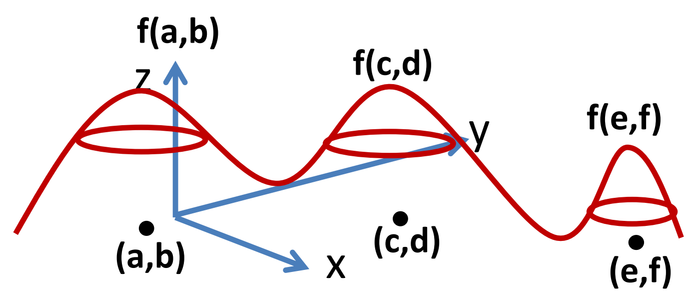


Definition: A function of two variables has a local minimum at (a,b) if f(a,b) ≤ f(x,y) when (x,y) is near (a,b)



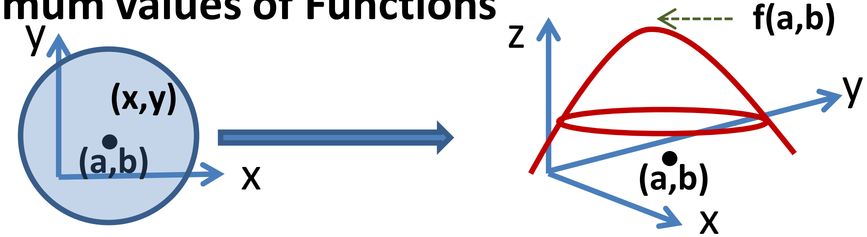
Definition: A local maximum at (a,b) is an absolute maximum if f(a,b) is greater than or equal to f at every other point in input space

N.B.: There can be more than one absolute maximum
---if they have the same value
f(a,b)=f(c,d) > f(e,f)



Section 14.7: Maximum and Minimum values of Functions

If f(x,y) has a local maximum at (a,b) and continuous partial first derivatives, then $f_x(a,b)=f_y(a,b) = 0$



input space

Output against input

We prove that $f_x(a,b)=0$

Step 1: Let g(x) =f(x,b).

Step 2: So g is a function of one variable, and has a maximum at x=a

Step 3: So that means that

$$\left.\frac{dg(x,b)}{dx}\right|_{x=a} = f_x(a,b) = 0$$ **DONE!**

[Same proof for showing that $f_y(a,b) = 0$]

Definition: Given f(x,y), a point (a,b) is a critical point if $f_x(a,b)=f_y(a,b) = 0$

Section 14.7: Maximum and Minimum values of Functions

Enough! Let's do some examples:

Example: Find the critical points of $f(x,y) = x^2 + y^2 - 2x - 6y + 14$

Solution: $f_x = 2x - 2$ so $f_x = 0$ when $x = 1$
$f_y = 2y - 6$ so $f_y = 0$ when $y = 3$ → **So there's one critical point at (1,3)**

Example: Find the critical points of $f(x,y) = (1/3)x^3 - (3/2)x^2 + 2x + (1/3)y^3 - (5/2)y^2 + 6y + 85$

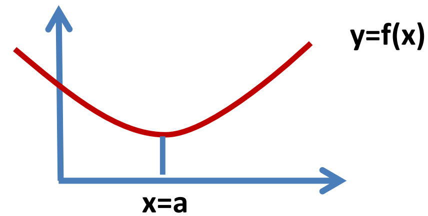
Solution: $f_x = x^2 - 3x + 2$ so $f_x = 0$ when $x = 1$ or $x = 2$
$f_y = y^2 - 5y + 6$ so $f_y = 0$ when $y = 2$ or $y = 3$ → **So there's four critical points (1,2), (2,2), (1,3) and (2,3)**

Fine. How can we tell if a critical point is a maximum or a minimum?

Section 14.7: Maximum and Minimum values of Functions

How can we tell if a critical point is a maximum or a minimum?

Good old-fashioned 1D:



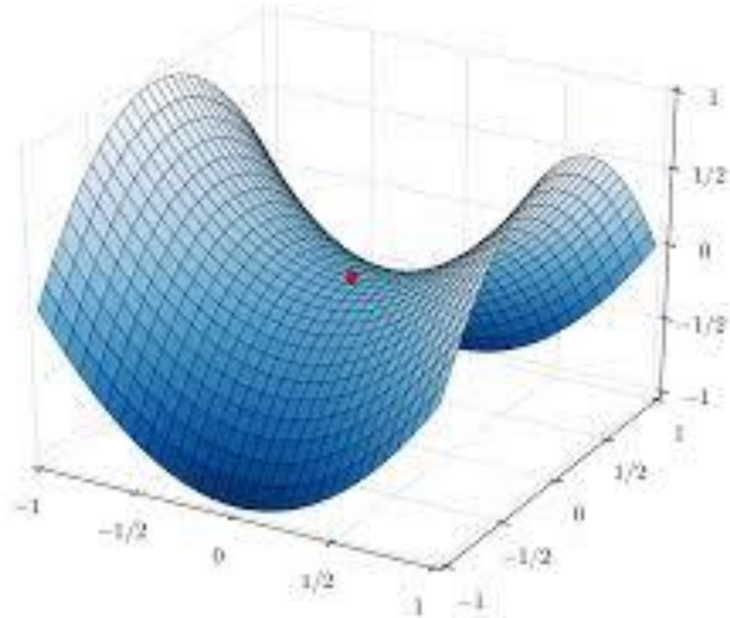
x=a is a critical point

$f_x(x=a) = 0$

$f_{xx}(x=a) > 0$

Unfortunately, it's not as simple in 2D:

The painful example: A saddle surface.



At the red critical point,
$f_x(\text{red\_point}) = f_y(\text{red\_point}) = 0$

But
$f_{xx}(\text{red\_point}) > 0$ and $f_{yy}(\text{red\_point}) < 0$

And this point is neither a minimum nor maximum!

We need a different test to determine if maximum or minimum

Section 14.7: Maximum and Minimum values of Functions

The "D" test to determine if maximum or minimum

Suppose the 2^{nd} partial derivatives of f(x,y) are continuous on a disk around (a,b)

Suppose $f_x(a,b) = f_y(a,b)=0$ (in other words, (a,b) is a critical point)

Let $D=D(a,b) = f_{xx}(a,b) * f_{yy}(a,b) - [f_{xy}(a,b)]^2$ **[Don't worry—no one can remember this!]**

Then

(a) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then (a,b) is a local minimum

(b) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then (a,b) is a local maximum

(c) If $D(a,b) < 0$ then it is neither a local minimum nor a local maximum