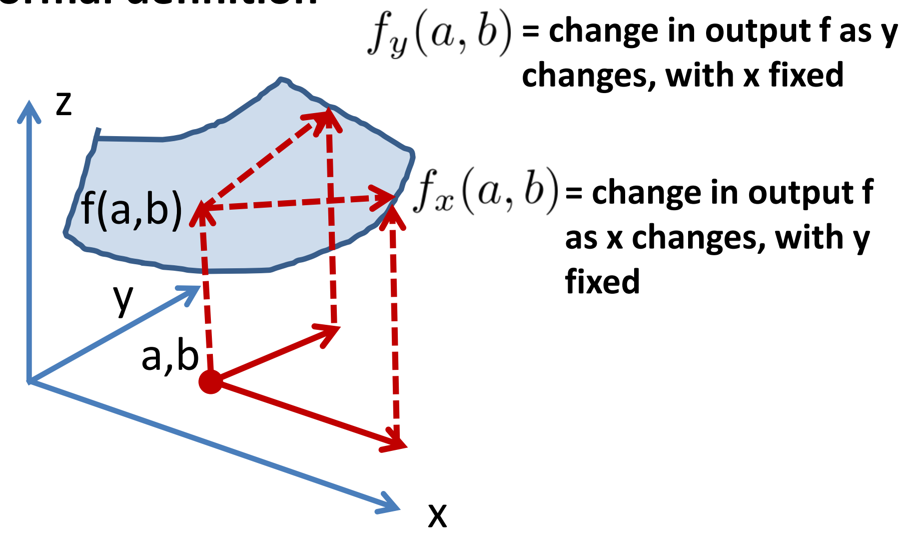
Section 14.3: The formal definition

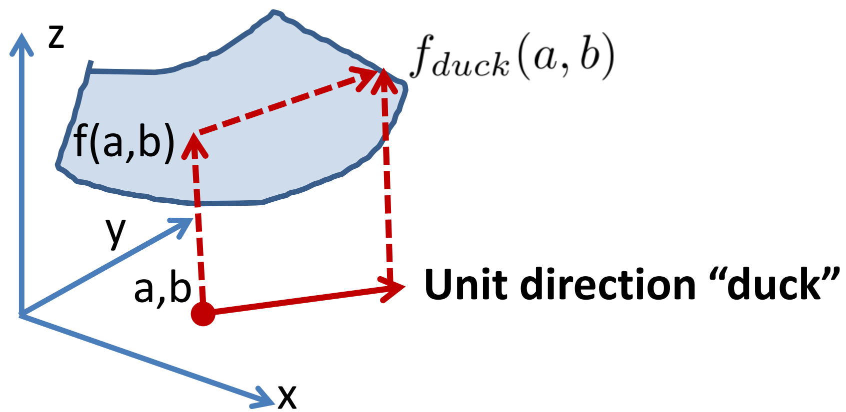
Let's review partial derivatives:

$$f_x(a,b) \equiv \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

$$f_y(a,b) \equiv \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$$

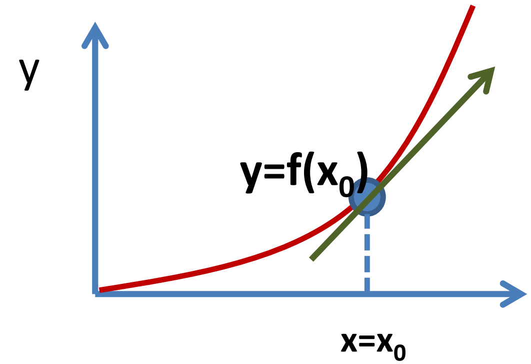


So, you should think of the partial derivative in the "duck" direction as how much the function changes as you move from (a,b) in the unit vector direction "duck"



Section 14.4: Tangent Planes and linear approximations

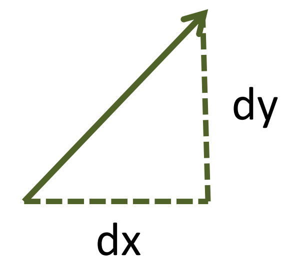
Let's recall 1D Calculus: y=f(x)



Tangent line with slope f'(x_0) going through the point (x_0,f(x_0))

Tangent line at (x_0,f(x_0)) touches the graph y=f(x) at only one point in near (x_0,f(x_0))

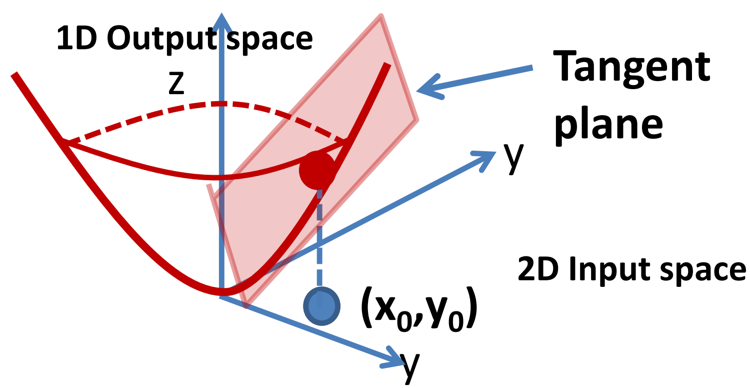
The tangent line is given by $y - f(x_0) = slope(x - x_0) = \left.\frac{df}{dx}\right|_{x_0} (x - x_0)$



Tangent vector = (dx,dy) = (1, dy/dx)= (1, f'(a))

We want to construct a similar idea for functions of two (or more variables):

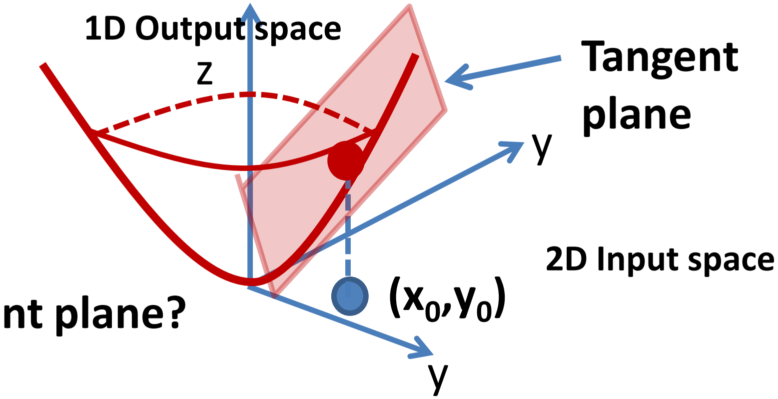
The Tangent Plane



Tangent plane at (x_0,y_0,f(x_0,y_0)) touches the graph z=f(x,y) at only one point is near (x_0,y_0,f(x_0,y_0))

Section 14.4: Tangent Planes and linear approximations

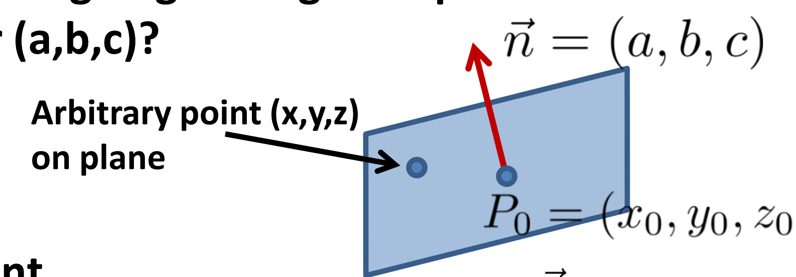
Tangent plane at $(x_0,y_0,f(x_0,y_0))$ touches the graph $z=f(x,y)$ at only one point is near $(x_0,y_0,f(x_0,y_0))$



How are we going to find an equation for the tangent plane?

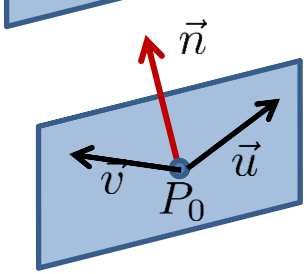
Idea #1! Do you remember we had a formula for a plane going through the point (x_0,y_0,z_0) with normal vector (a,b,c)?

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$



Realization 1: we know the point P_0 where the tangent plane touches the surface: $P_0 = (x_0,y_0,z_0)=(x_0,y_0,f(x_0,y_0))$

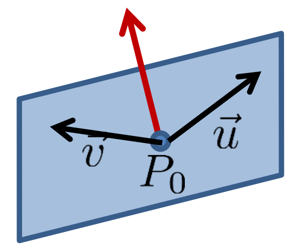
Realization 2: if we had two vectors $\vec{u}$ and $\vec{v}$ in the tangent plane, we could take their cross product to find the normal $\vec{n} = \vec{u} \times \vec{v}$



Section 14.4: Tangent Planes and linear approximations

Formula for a plane going through the point (x_0,y_0,z_0) with normal vector (a,b,c)

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

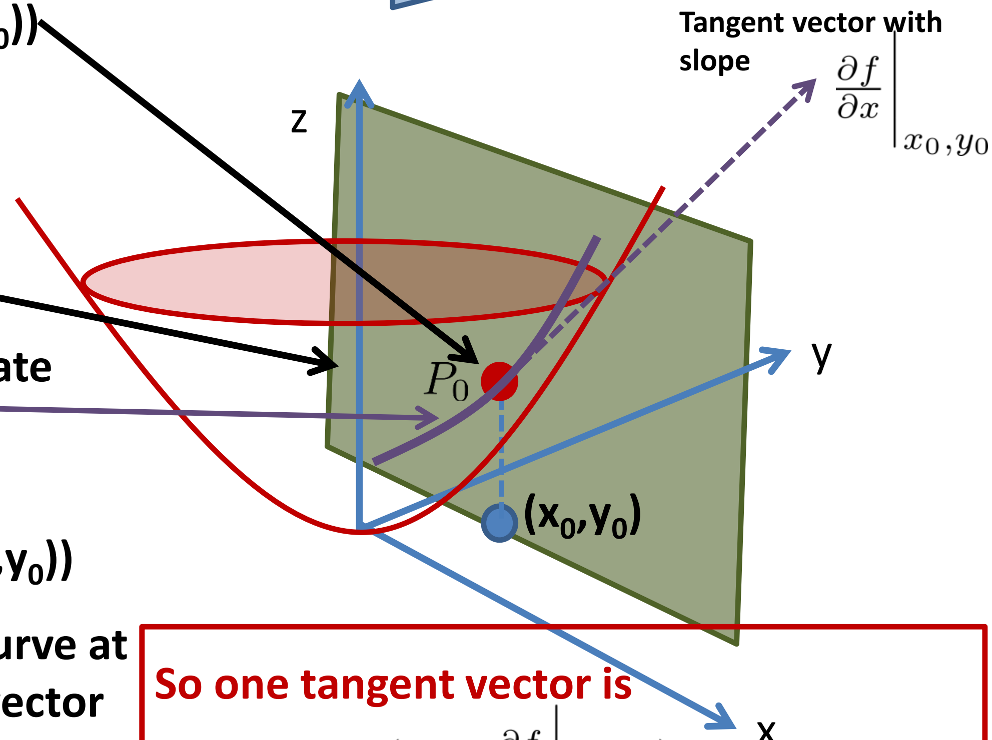


Tangent point is $P_0 = (x_0,y_0,z_0)=(x_0,y_0,f(x_0,y_0))$

How can we find two vectors in the tangent plane?

Idea #2: We can slice the graph of $z=f(x,y)$ with a plane $y=y_0$

This gives a purple curve whose y coordinate never changes and lies on the surface.



So this purple curve is the graph of $(x,y_0,f(x,y_0))$

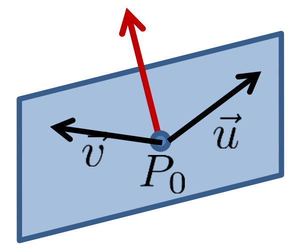
And the x partial derivative of this purple curve at (x_0,y_0) gives the slope $\left.\frac{df}{dx}\right|_{x_0,y_0}$ of the tangent vector at P_0 lying in the slicing plane

So one tangent vector is

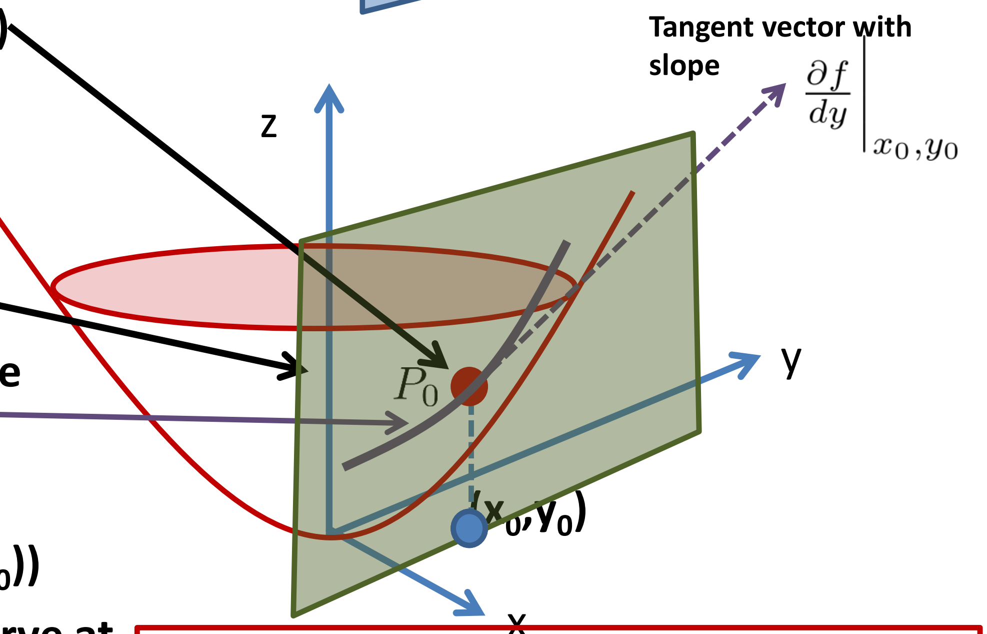
$$\vec{v} = (1, 0, \left.\frac{\partial f}{\partial x}\right|_{x_0,y_0})$$

Section 14.4: Tangent Planes and linear approximations

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$



Tangent point is $P_0 = (x_0,y_0,z_0)=(x_0,y_0,f(x_0,y_0))$



Obtain another tangent vector by slicing the graph of z=f(x,y) with a plane $x=x_0$

This gives a purple curve whose x coordinate never changes and lies on the surface.

So this purple curve is the graph of $(x_0,y,f(x,y_0))$

And the y partial derivative of this purple curve at (x_0,y_0) gives the slope $\left.\frac{df}{dy}\right|_{x_0,y_0}$ of the tangent vector at P_0 lying in the slicing plane

So one tangent vector is

$$\vec{v} = \left(0, 1, \left.\frac{\partial f}{\partial y}\right|_{x_0,y_0}\right)$$

Section 14.4: Tangent Planes and linear approximations

One tangent vector is $\vec{u} = (1, 0, \frac{\partial f}{\partial x}\Big|_{x_0, y_0})$ **One tangent vector is** $\vec{v} = (0, 1, \frac{\partial f}{\partial y}\Big|_{x_0, y_0})$

$$\vec{n} = (a, b, c) = \vec{u} \times \vec{v} = (1, 0, \frac{\partial f}{\partial x}) \times (0, 1, \frac{\partial f}{\partial y}) = (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1) \leftarrow \text{check this!}$$

Back to our formula for a plane:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

(I temporarily stopped writing $\Big|_{x_0, y_0}$)

Substitute everybody in:

$$-\frac{\partial f}{\partial x}(x - x_0) + -\frac{\partial f}{\partial y}(y - y_0) + 1(z - f(x_0, y_0)) = 0$$

Solve for z (remembering that $z_0 = f(x_0,y_0)$):

$$z - f(x_0, y_0) = \frac{\partial f}{\partial x}\Big|_{x_0, y_0}(x - x_0) + \frac{\partial f}{\partial y}\Big|_{x_0, y_0}(y - y_0)$$

Formula for the tangent plane

Section 14.4: Tangent Planes and linear approximations

Formula for the tangent plane

$$z - f(x_0, y_0) = \left.\frac{\partial f}{\partial x}\right|_{x_0,y_0} (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{x_0,y_0} (y - y_0)$$

Notice how much it looks like our formula for the slope of a tangent line:

$$y - f(x_0) = slope(x - x_0) = \left.\frac{df}{dx}\right|_{x_0} (x - x_0)$$

Example: Find the equation for the plane tangent to $z= 2x^2 + y^2$ at the input point $x_0=1$ $y_0=1$

Solution: Step 1: find the point on the surface at the input $x_0=1$ $y_0=1$

$$z_0 = f(x_0,y_0)=f(1,1) = 2\,(1)^2 + 1^2 = 3$$

Step 2: find the partial derivatives at the input point:

$f_x = 4x$ so at input $x_0=1$ $y_0=1$ $f_x=4$

$f_y = 2y$ so at input $x_0=1$ $y_0=1$ $f_y=2$

Step 3: put them into your equation for the tangent plane: $z-3 = 4\,(x-1) + 2\,(y-1)$

Section 14.4: Tangent Planes and linear approximations

Why stop there?

1d: Slope of a tangent line

$$f(x) = f(a) + \left.\frac{df}{dx}\right|_a (x-a) \leftarrow \text{Equation for Line tangent to } f(a)$$

2d: Formula for the tangent plane

$$f(x,y) = f(a,b) + \left.\frac{\partial f}{\partial x}\right|_{a,b}(x-a) + \left.\frac{\partial f}{\partial y}\right|_{a,b}(y-b) \leftarrow \text{Equation for plane tangent to } f(a,b)$$

3d: Formula for the tangent "hyperplane"

$$f(x,y,z) = f(a,b,c) + \left.\frac{\partial f}{\partial x}\right|_{a,b,c}(x-a) + \left.\frac{\partial f}{\partial y}\right|_{a,b,c}(y-b) + \left.\frac{\partial f}{\partial z}\right|_{a,b,c}(z-c)$$

$$\text{Equation for hyperplane tangent to } f(a,b,c)$$

"n"d: Formula for the tangent "hyperplane"

$$f(x_1, ...x_n) = f(a_1, a_2, ...a_n) + \left.\frac{\partial f}{\partial x_1}\right|_{a_1,a_2,...,a_n}(x_1 - a_1) + \left.\frac{\partial f}{\partial x_2}\right|_{a_1,a_2,...,a_n}(x_2 - a_2) + ... + \left.\frac{\partial f}{\partial x_n}\right|_{a_1,a_2,...,a_n}(x_n - a_n)$$

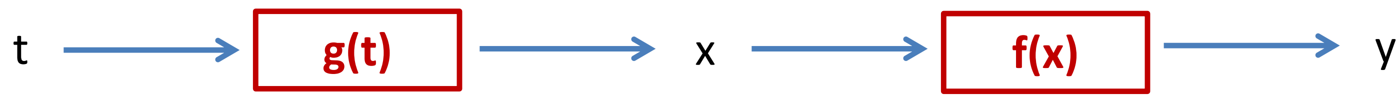
$$\text{Equation for hyperplane tangent to } f(a_1, a_2, ..., a_n)$$

(No—I can't draw it!)

Section 14.5: The Chain Rule

This is beautiful— but it requires some conceptual thinking. Let's go!

Begin by recalling the 1D chain rule:

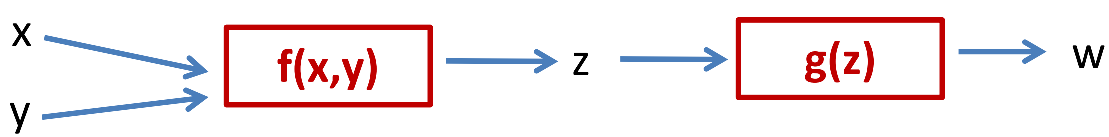


Then if we ask "how does the output y change as we change the input t",

Then we are asking for $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$

What does mean when more than one variable is involved as input?

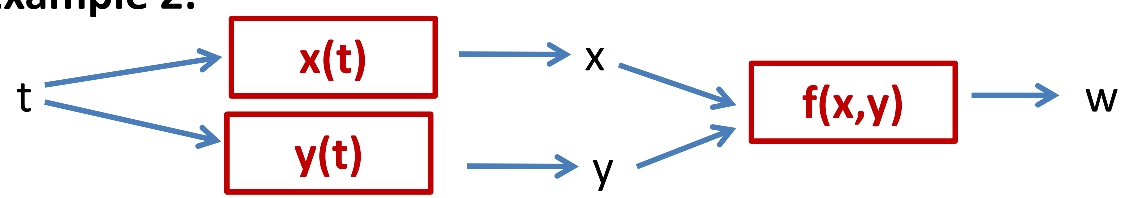
Example 1:



What is $\frac{\partial w}{\partial x}$ **?**
What is $\frac{\partial w}{\partial y}$ **?**

Example 2:

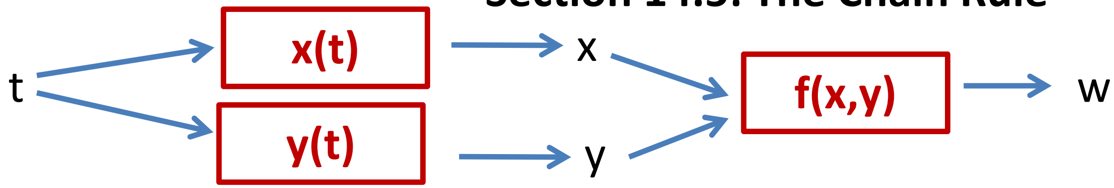
Many different diagrams to draw!



What is dw/dt?

The book has many examples of diagrams. They are *all* the same thing!

Section 14.5: The Chain Rule



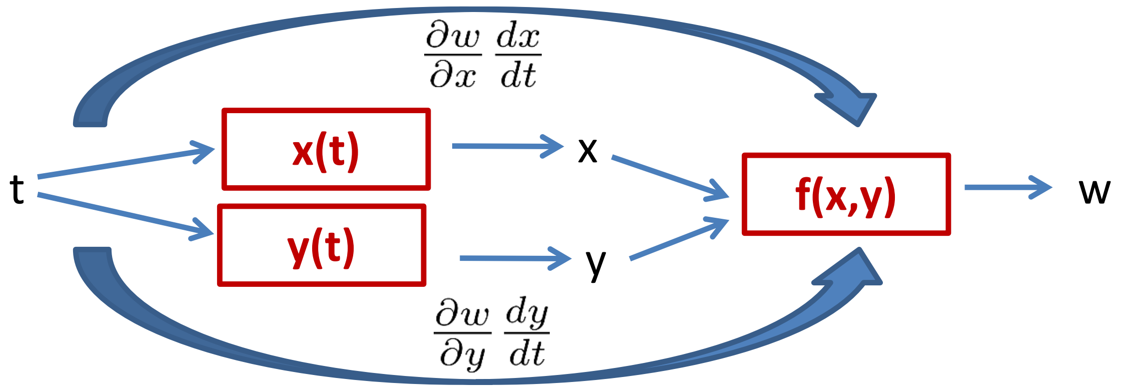
What is dw/dt?

Claim: $\frac{dw}{dt} = \text{Change in output w as input t changes} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$

First question: **Why do I write "d" on the left but partial signs ∂ on the right?**

Answer: because w depends on only one variable "t", hence it's a simple derivative

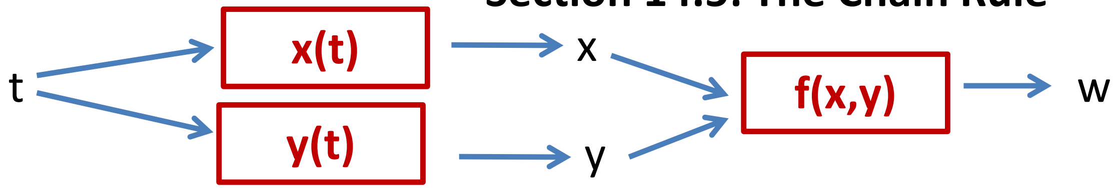
Second question: **What is the meaning of the two terms that are added?:**



Answer: they represent contributions from two different paths through the diagram

Third question: **Why do we add the contributions from the two paths? First, an example**

Section 14.5: The Chain Rule



What is dw/dt?

Claim: $\frac{dw}{dt} = \text{Change in output w as input t changes} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$

Before proving this, let's do an example

If $w(x,y)=x^3y + 3xy^4$ and $x=\sin(2t)$ and $y=\cos(t)$, find $\frac{dw}{dt}$

Solution:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

$$= (3x^2y + 3y^4)(2\cos 2t) + (x^3 + 12xy^3)(-\sin t)$$

$$= \left[3(\sin 2t)^2(\cos t) + 3(\cos t)^4\right](2\cos 2t)$$

$$+ \left[(\sin 2t)^3 + 12(\sin 2t)(\cos t)^3\right](-\sin t)$$

Section 14.5: The Chain Rule

Again: $$\frac{dw}{dt} = \left[3(\sin 2t)^2(\cos t) + 3(\cos t)^4\right](2\cos 2t) + \left[(\sin 2t)^3 + 12(\sin 2t)(\cos t)^3\right](-\sin t)$$

Note—you could do this the "old way"

If w(x,y)=x^{3}y + 3xy^4 and x=sin(2t) and y=cos(t), find $\frac{dw}{dt}$

Solution (old way)--substitute:

$$z(x,y) = (\sin 2t)^3 \cos t + 3\sin 2t(\cos t)^4$$

$$\frac{dw}{dt} = \left[3(\sin 2t)^2(2\cos 2t)\right]\cos t + (\sin(2t))^3(-\sin t)$$

$$3(2\cos 2t)(\cos t)^4 + 3\sin 2t\left[4(\cos t)^3\right](-\sin t)$$

They are the same! And the new way is much easier in complicated cases

Section 14.5: The Chain Rule

Your turn: $w(x,y)=xy^3 - x^2y$ $x(t) = t^2 + 1$ and $y(t) = t^2-1$ Find $\frac{dw}{dt}$

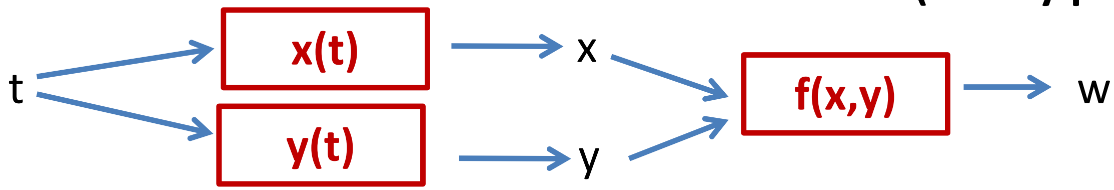
Solution: $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$

$$= (y^3 - 2xy)(2t) + (3xy^2 - x^2)(2t)$$

$$= ((t^2-1)^3 - 2(t^2+1)(t^2-1))(2t)$$

$$+(3(t^2+1)(t^2-1)^2 - (t^2+1)^2)(2t)$$

Section 14.5: The Chain Rule (a flaky proof)



What is dw/dt?

Claim: $\frac{dw}{dt} = \text{Change in output w as input t changes} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$

Proof: Step 1: Recall our tangent approximation plane to f(x,y) at an input point (a,b):

$$f(x,y) = f(a,b) + \frac{\partial f}{\partial x}\bigg|_{a,b}(x-a) + \frac{\partial f}{\partial y}\bigg|_{a,b}(y-b) \leftarrow \text{Equation for plane tangent to } f(a,b)$$

Step 2: So we can call w= f(x,y), Δx = x-a, Δy = y-b

Step 3: Rewriting, we have

$$w(x,y) = w(a,b) + \frac{\partial w}{\partial x}(\triangle x) + \frac{\partial w}{\partial y}(\triangle y)$$
$$w(x,y) - w(a,b) = \frac{\partial w}{\partial x}(\triangle x) + \frac{\partial w}{\partial y}(\triangle y)$$

$$\Rightarrow \triangle w = \frac{\partial w}{\partial x}(\triangle x) + \frac{\partial w}{\partial y}(\triangle y)$$

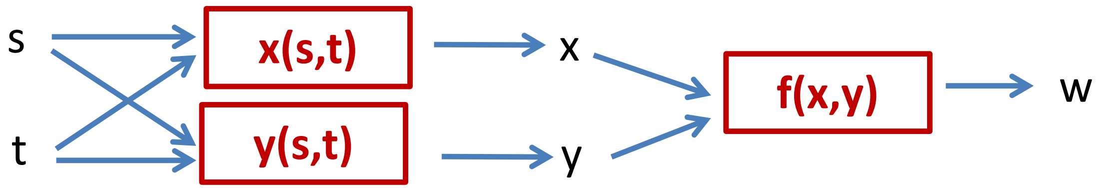
Step 4: Divide both sides by Δt $\frac{\triangle w}{\triangle t} = \frac{\partial w}{\partial x}\frac{\triangle x}{\triangle t} + \frac{\partial w}{\partial y}\frac{\triangle y}{\triangle t}$

Step 5: Take the limit as Δt goes to zero: $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$

This is not much of a proof—but gives the idea of why the terms are added

Section 14.5: The Chain Rule (onwards!)

A more complicated case



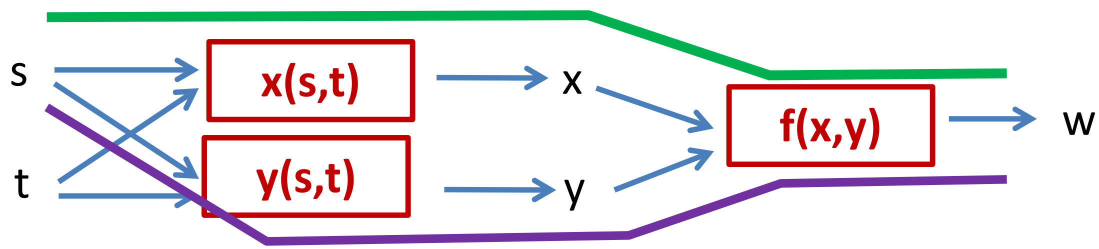
What is $\frac{\partial w}{\partial s}$? $\frac{\partial w}{\partial t}$?

Remember:

$\frac{\partial w}{\partial s}$ **is "how output w changes as input s changes, holding all other inputs constant"**

Step 1: Find all paths through the diagram:

for $\frac{\partial w}{\partial s}$



Path contributions

path (green) $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s}$ **path** (purple) $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$

Add the paths $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$

Similarly $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$

Section 14.5: The Chain Rule (your turn)

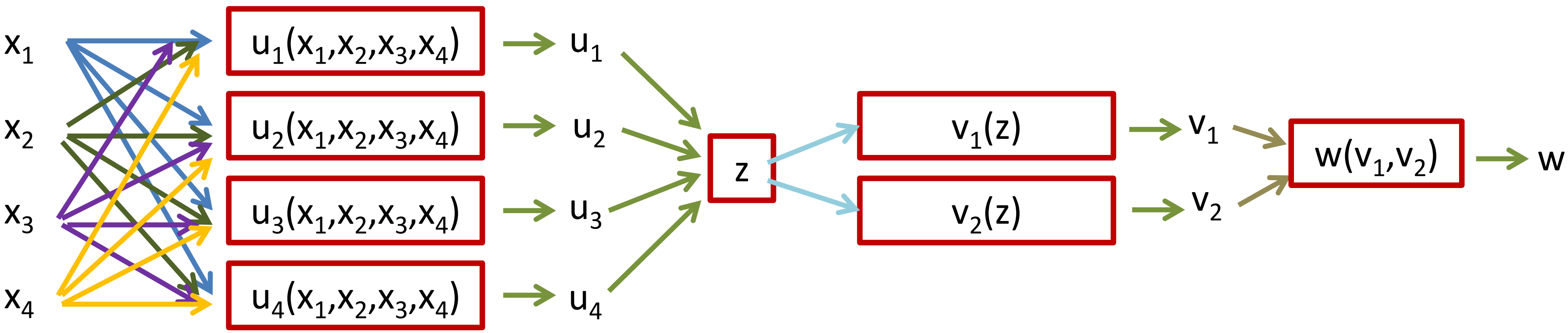
If $w(x,y) = e^x \sin y$ and $x = st^2$ and $y = s^2t$ find $\frac{\partial w}{\partial s}$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial w}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st)$$

$$= (e^{st^2} \sin(s^2t))(t^2) + (e^{st^2} \cos(s^2t))(2st)$$

Section 14.5: The Chain Rule (one more for you)



What is $\dfrac{\partial w}{\partial x_3}$ **?**

$$\frac{\partial w}{\partial x_3} = \frac{dw}{dz}\frac{\partial z}{\partial x_3}$$

Again—evaluate all possible paths:

$$= \left[\frac{\partial w}{\partial v_1}\frac{dv_1}{dz} + \frac{\partial w}{\partial v_2}\frac{dv_2}{dz}\right]\left[\frac{\partial z}{\partial u_1}\frac{\partial u_1}{\partial x_3} + \frac{\partial z}{\partial u_2}\frac{\partial u_2}{\partial x_3} + \frac{\partial z}{\partial u_3}\frac{\partial u_3}{\partial x_3} + \frac{\partial z}{\partial u_4}\frac{\partial u_4}{\partial x_3}\right]$$